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THE LOGARITHMIC POTENTIAL

DISCONTINUOUS DIRICHLET AND NEUMANN PROBLEMS

BY

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TO

VITO VOLTERRA

PREFACE

This small treatise is an outgrowth of a study of Stieltjes integrals and potential theory which the author published in the 1920 volume of the Rice Institute Pamphlet, and a needed revision and development of the last part of that essay in the direction indicated by three notes which appeared in 1923, in the Comptes rendus des séances de l'Academie des Sciences. Two of these were written in conjunction with my colleague, Professor H. E. Bray. The work gives a unified treatment of the basis of the theory of Laplace's equation in two dimensions, suitable, it is hoped, for graduate students of a moderate degree of advancement, and is intended to be of service in the development of the theory of partial differential equations of elliptic type. These developments are generating a compound of two of the most important elements of modern analysis—the concepts of Lebesgue on the one hand, and of Volterra on the other.

An earlier form of part of the treatise was given in lectures at the Rice Institute in the academic year 1924–25, in connection with a course in the theory of functions of a real variable, and at the University of Chicago during the Summer Quarter of 1925. Chapter VII furnished the substance of an invited discourse at the meeting of the Southwestern Section of the American Mathematical Society in November, 1926.

The author is much indebted to Professor O. D. Kellogg, who has seen a large portion of the manuscript, and aided with kindly criticism, to Professor Bray, who has read the proof sheets, and, finally, to the American Mathematical Society, through whose generosity the publication is possible.

Houston, Texas.
June, 1927.

Griffith C. Evans.

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CHAPTER I

PRELIMINARY CONCEPTS
STIELTJES INTEGRALS AND FOURIER SERIES

1. Functions of limited variation. In the chapters which follow, the Dirichlet and Neumann problems are recast under general points of view which derive, as did earlier formulations of the problems, from physical considerations. These problems are to be discussed in relation to conformal transformations and the most general distributions of matter, simply or in doublets, on the boundary of a general simply or finitely connected open region bounded by circles. In this way new classes of boundary conditions arise, and the appropriate boundary value problems may be solved. Thus for the old problems some new results, as well as some familiar ones, appear as special cases.

The principal instrument for the investigation of harmonic functions, from this point of view, will be the *Stieltjes integral.* This integral refers fundamentally to functions *of limited variation.* For the convenience of the reader some important properties of the relation between these two concepts will be briefly summarized, and one or two theorems obtained.

Let $f(x)$ be defined for every x in the closed interval (a, b), and let this interval be divided into a finite number n of subintervals by points $x_1, x_2, \cdots, x_{n-1}$, writing $x_0 = a$, $x_n = b$ for convenience. If the quantity

$$\sum_{0}^{n-1}{}_i\, |f(x_{i+1}) - f(x_i)|, \qquad x_0 < x_1 < x_2 \cdots < x_n,$$

is bounded, $\leqq N$, for all such positions of the x_i in (a, b) and all n, the function is said to be *of limited variation* in (a, b). The least value of N which will satisfy this condition is said to be the *total variation* of $f(x)$ in (a, b), and is designated by T.

In any closed interval (a, x), $a < x \leqq b$, let us choose a set of non-overlapping subintervals (x_1', x_1''), (x_2', x_2''), $\cdots$.

finite in number, which add together into the whole or less than the whole of (a, x). The upper bound, for all such choices, of the quantity

$$\sum_k (f(x_k'') - f(x_k'))$$

exists, and we designate it by $\varphi(x)$ or φ_{ax}; that is to say, $\varphi(x)$ is the smallest number which is not exceeded by any possible value of the given sum. Let us definitely admit a single point as a special case of an interval, namely one whose end points coincide. We see then directly that $\varphi(x) \geqq 0$, and again that $\varphi(x)$ is a non-decreasing function of x. It is called the *positive variation function* of $f(x)$.

Exercise. Show that $\varphi(x)$ has the same value and the same properties even if we do not admit intervals of zero length. These properties may be demonstrated by proving that given ε arbitrarily small we can find in any interval (x_1, x_2) some subinterval (x', x'') for which $|f(x'') - f(x')| < \varepsilon$.

In order to complete the definition of $\varphi(x)$ we must assign it a value at $x = a$; we assign it there the value 0.

It is interesting to note that if X is some value intermediate between a and x we have

$$\varphi_{ax} = \varphi_{aX} + \varphi_{Xx}$$

or

$$\varphi(x) = \varphi(X) + \varphi_{Xx}.$$

For we can form a sum for the interval (a, x) as near to φ_{ax} as we desire; if X is an interior point of a subinterval the sum is not changed if we insert X as the end of one interval and the beginning of the next; thus in any case we can form partial sums which relate to φ_{aX} and φ_{Xx} respectively, yet cannot exceed them. Hence $\varphi_{ax} \leqq \varphi_{aX} + \varphi_{Xx}$. Also we can form partial sums for (a, X) and (X, x) as near respectively to φ_{aX} and φ_{Xx} as we desire; the sum total will be $\leqq \varphi_{ax}$. Hence $\varphi_{ax} \geqq \varphi_{aX} + \varphi_{Xx}$, from which the conclusion follows.

We define another function of x, which is a non-increasing function:

$$\psi(x) = f(x) - f(a) - \varphi(x).$$

In fact, if x_1, x_2 are two values of x, with $x_2 > x_1$, we have

$$\psi(x_2) - \psi(x_1) = f(x_2) - f(x_1) - \{\varphi(x_2) - \varphi(x_1)\}.$$

But $\varphi(x_2) - \varphi(x_1) = \varphi_{x_1 x_2} \geqq f(x_2) - f(x_1)$. Hence $\psi(x_2) - \psi(x_1) \leqq 0$, and $\psi(x)$ is a non-increasing function of x.

The function $\psi(x)$ is called the *negative variation function* of $f(x)$. Moreover by the definition of $\psi(x)$ we have

$$f(x) = f(a) + \varphi(x) + \psi(x).$$

In other words, a function of limited variation in the closed interval (a, b) can be written as the difference of two non-decreasing functions of x. The converse of this statement is obviously true.

The function $t(x) = \varphi(x) - \psi(x)$ is again a non-decreasing function of x, being the sum of two such functions, and is called the *total variation function* of $f(x)$.

EXERCISE. Show that definitions of $\psi(x)$ and $t(x)$ analogous to that of $\varphi(x)$ may be given, and that $t(b) = T$.

A non-decreasing function approaches a limiting value as x approaches a value α from the left, and also as x approaches α from the right. Hence the same property holds for the difference between two such functions, that is, for any function of limited variation. The two values, the limit from the left and the limit from the right, need not be equal, or equal to the value $f(\alpha)$, since $f(x)$ need not be continuous at α. It is therefore convenient to have symbols for these values and write $f(\alpha + 0)$ for the quantity $\lim\limits_{\substack{x=\alpha \\ x>\alpha}} f(x)$, and $f(\alpha - 0)$ for the quantity $\lim\limits_{\substack{x=\alpha \\ x<\alpha}} f(x)$, respectively. If $\alpha = 0$, the symbols $f(0+)$ and $f(0-)$ are used. The statement we have just made then amounts to saying that if $f(x)$ is of limited variation, $f(x+0)$ and $f(x-0)$ exist if $a < x < b$; and also $f(a+0)$ and $f(b-0)$ exist.

An important property of a function of limited variation is that the aggregate of its points of discontinuity must be denumerable. To prove this fact it is sufficient to consider

a non-decreasing function, since the discontinuities of a function of limited variation will also be discontinuities of its total variation function. But for such a function the number of points x where $f(x+0)-f(x)$ or $f(x)-f(x-0)$ is $\geqq T/2$ is finite; also the number of points such that these jumps are $< T/2$ but $\geqq T/4$; also the number of points for which these jumps are $< T/4$ but $\geqq T/8$, etc. In such a classification however every point of discontinuity is ultimately included, since a point where both $f(x+0)-f(x)$ and $f(x)-f(x-0)$ are both zero is a point of continuity.

2. **Continuation.** We have discussed the function of limited variation with respect to a closed interval (a, b). It is convenient however to be able to consider the same sort of situation with respect to an open interval. That is we say that $f(x)$ is of limited variation if

$$\sum |f(x_k'')-f(x_k')| \leqq N,$$

N being some constant, no matter how the subintervals (x_k', x_k'') are chosen in the open interval (a, b). We define the positive variation function as before; it is again a non-decreasing function of x, with $\varphi(a+0)=0$; moreover $\varphi(x)=\varphi(X)+\varphi_{Xx}$ if $a<X<x<b$. Further let $\psi_1(x)=f(x)-\varphi(x)$; this is a non-increasing function. Hence $\psi_1(a+0)$ exists. We therefore define the negative variation function as

$$\psi(x) = \psi_1(x)-\psi_1(a+0),$$

so that $\psi(a+0)=0$. We define the total variation function as

$$t(x) = \varphi(x)-\psi(x),$$

and we have also $t(a+0)=0$. Whether or not for completeness we define $\varphi(a)$, $\psi(a)$ and $t(a)$ as 0 is immaterial, since a is outside the open interval. Finally, we have

$$f(x) = \varphi(x)+\psi(x)+\psi_1(a+0),$$

so that $f(a+0)=\psi_1(a+0)$. The quantities $f(a+0)$ and $f(b-0)$ are thus seen to be determinate, although $f(a)$ and $f(b)$ are not defined. The quantity $t(b-0)-t(a+0)$ is seen to

be the least value possible of the N above, and may be spoken of as the total variation T' of $f(x)$ for the open interval (a, b).

It is obvious that a function which is of limited variation on an open interval may be extended so that it will be of limited variation on the corresponding closed interval. This is a particular case of the following theorem.

THEOREM 1. *Let E be a set dense in (a, b), not necessarily including the end points a, b, and let $f(x)$ be of limited variation on E. Then $f(x)$ may be defined on the complementary set CE so as to be of limited variation on the closed interval (a, b).*

Our hypothesis about E is that every subinterval of (a, b) not of zero length contains at least one point of E. Let x be any value $a < x \leq b$. Consider a finite number of non-overlapping intervals (x_i', x_i''), $a \leq x_i' \leq x_i'' \leq x$, of which the ends belong to E, and consider the sums $\sum |f(x_i'') - f(x_i')|$, $\sum \{f(x_i'') - f(x_i')\}$. If the former sum has an upper bound T, when $x = b$, $f(x)$ is said to be of limited variation on E. In this case, given x, the latter sum has an upper bound $\varphi(x) = \varphi_{ax}$ which is not negative. The definition of $\varphi(x)$ is completed by writing $\varphi(a) = 0$.

We notice that $\varphi(x)$ is a non-decreasing function of x, and that if x is a point of CE we have $\varphi(x) = \varphi(x-0)$. Moreover if $x > X$ and X is in E, we have $\varphi(x) = \varphi(X) + \varphi_{Xx}$.

EXERCISE. The reader may prove that if X is in CE we have $\varphi(x) = \varphi(X+0) + \varphi_{Xx}$.

We define also $\psi(x)$ on the points of E,

$$\psi(x) = f(x) - \varphi(x) + \text{const.}$$

where the constant is to be assigned. The reasoning previously employed shows that on E the function $\psi(x)$ is a non-increasing function. Hence the upper bound of $\psi(x)$ on E is $\psi(a)$ if a is in E; otherwise it is $\psi(a+0)$. But in either case we may choose the arbitrary constant so that this upper bound is zero.

We may now proceed to extend the definition of $\psi(x)$ by defining a function $\beta(x)$ for the function $-\psi(x)$ in the same

way as before we defined $\varphi(x)$ for $f(x)$. But since on the points of E, $-\psi(x)$ is a non-decreasing function with lower bound zero, it follows that $\beta(x) = -\psi(x)$ on E. In order to complete the definition of $\psi(x)$ we write for the points of CE

$$\psi(x) = -\beta(x).$$

In particular, since $\varphi(x)$ and $\psi(x)$ are monotonic, the quantities $\varphi(a+0)$, $\psi(a+0)$, $\varphi(b-0)$, $\psi(b-0)$ and the quantities $\varphi(x \pm 0)$, $\psi(x \pm 0)$ for $a < x < b$ are all defined and equal to the values which are obtained for them by considering merely the points of E; moreover for points of CE we have $\varphi(x-0) = \varphi(x)$, $\psi(x-0) = \psi(x)$.

It follows then that $f(a+0)$ is defined by means of the points of E, and if a is not a point of E we are at liberty to define $f(a) = f(a+0)$. To define $f(x)$ on the points of CE we write

$$f(x) = f(a) + \varphi(x) + \psi(x).$$

The function $f(x)$, so extended, is of limited variation on (a, b); moreover $f(x+0)$ and $f(x-0)$ have the values which are defined merely by the points of E, and also $f(x-0) = f(x)$ on the points of CE. The functions $\varphi(x)$ and $\psi(x)$ are evidently the positive and negative variation functions respectively for $f(x)$, and the function $t(x) = \varphi(x) - \psi(x)$ is the total variation function.

If $g(x)$, of limited variation, is any other extension of $f(x)$ which agrees with the common value of $f(x+0)$ and $f(x-0)$, whenever these have a common value as determined by points of E, then $g(x) = f(x)$ throughout, except at the points of discontinuity of $f(x)$, which of course, constitute at most a denumerable infinity of points in E and CE; for $f(x+0)$ and $f(x-0)$ defined by means of E are the same as when defined by means of all the points of (a, b), and therefore if x is a point of continuity or an unnecessary discontinuity of $f(x)$, the function $g(x)$ will be $f(x-0)$.

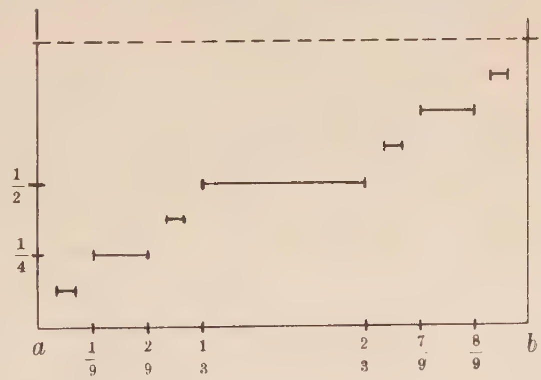
We have also immediately the following corollaries, of which the proofs are obvious.

COROLLARY 1. Any function of limited variation on (a, b) which is an extension of $f(x)$ on the points of E agrees with the extension above defined except at a denumerable infinity of points; if the discontinuities are made regular on (a, b) the extension is uniquely determined (except at a and b, if a, b are members of CE).*

COROLLARY 2. The proofs of these theorems may be applied at once to the case where the set of points E is dense on the circumference of a circle, and f is given, of limited variation, on E.† It is evident then that f may be so defined as to be of limited variation on the whole circumference, that any two such extensions differ only on a denumerable set, and if the discontinuities are made regular the function is uniquely determined on the circumference.

COROLLARY 3. If $f(x)$ is the difference of two not negative *bounded* functions on E it is of limited variation on E.

EXERCISE. The figure illustrates a process of arriving at a function of limited variation (non-decreasing) which is continuous, whose derivative is zero for points in intervals whose lengths add together into the total length of (a, b), and yet which is not constant in (a, b). The reader may demonstrate these facts. This type of example is due to Cantor, Vitali and Borel.



3. Integrals with respect to a function of limited

* The function $f(x)$ has a regular discontinuity at x_0 if $f(x_0) = \frac{1}{2}\{f(x_0+0)+f(x_0-0)\}$.

† E is such that every circular arc contains at least one point of E; f is a function of the central angle θ or of some variable which is a continuous monotonic function of θ.

variation. We turn now to the definition of our integral. Let $u(x)$ be a bounded function and $f(x)$ a monotonic function, say non-decreasing, in the closed interval (a, b). Let M_i be the upper, and m_i the lower, bound of $u(x)$ in the sub-interval (x_i, x_{i+1}), and form the sums

$$(1)\qquad \begin{aligned} S_n &= \sum_0^{n-1} M_i \{f(x_{i+1}) - f(x_i)\}, \\ s_n &= \sum_0^{n-1} m_i \{f(x_{i+1}) - f(x_i)\}. \end{aligned}$$

Evidently S_n cannot be increased by adjoining new points of subdivision, nor can s_n be decreased; further $S_n \geqq s_n$. The lower bound S of S_n is called the superior integral, the upper bound s of s_n is called the inferior integral, and if the two bounds are the same the common value, $S = s$, is called the Stieltjes, (or sometimes the Riemann-Stieltjes) integral. It is denoted by the symbol

$$\int_a^b u(x)\, df(x).$$

If $u(x)$ is continuous, *the Stieltjes integral exists.* In fact, since $f(x)$ is non-decreasing, the classical method of Jordan for the Riemann integral applies precisely to this case. Moreover, the law of the mean is justified, that

$$(2)\qquad \int_a^b u(x)\, df(x) = u(\xi)[f(b) - f(a)], \qquad a \leqq \xi \leqq b.$$

Exercise. In the inequality $a \leqq \xi \leqq b$, the equal signs may not be omitted. The reader may find an example to verify this fact.

Exercise. Show that

$$\left| \int_a^b u(x)\, df(x) \right| \leqq T \max |u(x)|.$$

If $u(x)$ is continuous and $f(x)$ of bounded variation, not necessarily monotonic, we *define*

$$(3)\qquad \int_a^b u(x)\, df(x) = \int_a^b u(x)\, d\varphi(x) + \int_a^b u(x)\, d\psi(x)$$

where φ and ψ are the positive and negative variations of $f(x)$. If we insert the points of subdivision $a = x_0$,

$x_1, \cdots, x_n = b$, with $x_{i+1} - x_i < \delta$, and let x_i' be a point in the interval $x_i \leqq x_i' \leqq x_{i+1}$, we have

$$(4) \qquad \left| \int_a^b u(x)\, df(x) - \sum_0^{n-1} u(x_i')\{f(x_{i+1}) - f(x_i)\} \right| \leqq T\omega_\delta$$

where ω_δ is the oscillation of $u(x)$ in an interval of length δ in (a, b). Hence, in particular,

$$(5) \qquad \lim_{\delta=0} \sum_{i=0}^{n-1} u(x_i')\{f(x_{i+1}) - f(x_i)\} = \int_a^b u(x)\, df(x),$$

a result which conforms to a definition of the Stieltjes integral according to the method of Cauchy in the ordinary case.

We may take equation (5) as an alternative definition of the Stieltjes integral, entirely suitable for our purposes.

A limit of this sort may exist even if $f(x)$ is not of limited variation and $u(x)$ continuous; in fact it always does exist if the roles of the two functions are reversed, $u(x)$ being of limited variation and $f(x)$ continuons. This is demonstrated by means of Stieltjes's celebrated integration by parts, which we now proceed to carry through.

We have, assuming $f(x)$ to be of limited variation, and $u(x)$ continuous, taking x_i' in the interval $x_i \leqq x_i' \leqq x_{i+1}$, the equality

$$\begin{aligned} &\sum_{k=0}^{n-1} f(x_k')\{u(x_{k+1}) - u(x_k)\} - f(x_n)\,u(x_n) + f(x_0)\,u(x_0) \\ &\quad = u(x_0)\{f(x_0) - f(x_0')\} + \sum_{k=1}^{n-1} u(x_k)\{f(x_{k-1}') - f(x_k')\} \\ &\qquad\qquad + u(x_n)\{f(x_{n-1}') - f(x_n)\} \\ &\quad = -\sum_{k=0}^{n} u(x_k)\{f(x_k') - f(x_{k-1}')\}, \end{aligned}$$

where for convenience we write $x_{-1}' = x_0 = a$, $x_n' = x_n = b$. Since as $\delta \doteq 0$ the limit of the summation in the last member exists, and is a Stieltjes integral, the same applies to the first summation, whose limit gives a Cauchy definition of the other Stieltjes integral; hence, if we take the limit,

$$(6) \qquad \int_a^b f(x)\, du(x) = f(x)\,u(x)\Big]_a^b - \int_a^b u(x)\, df(x).$$

If $f(x)$ and $u(x)$ are periodic, wich period $b-a$, or if they are defined on the circumference of a circle, x being a continuous monotonic function of θ, we have $\int f(x)\,du(x) = -\int u(x)\,df(x)$.

As a concluding paragraph in this section, we define the Stieltjes integral for an open interval (a, b) and thus interpret the symbol $\int_{a+0}^{b-0} u(x)\,df(x)$. We take $f(x)$ of limited variation in the open interval (a, b). In the sums S_n and s_n we replace $f(a) = f(x_0)$ by $f(a+0) = f(x_0+0)$ and $f(b) = f(x_n)$ by $f(b-0) = f(x_n-0)$, and say that the Stieltjes integral on the open interval exists if $S = s$, and we take that common value as its value. The integral exists if $u(x)$ is *uniformly* continuous in the open interval, and in this case all the formulae previously given may be carried over merely changing $f(x_0)$ to $f(a+0)$ and $f(x_n)$ to $f(b-0)$ throughout. *In the law of the mean, ξ lies actually between a and b.* In particular, we have the integration by parts

$$(7) \quad \int_{a+0}^{b-0} f(x)\,du(x) = f(x)u(x)\Big]_{a+0}^{b-0} - \int_{a+0}^{b-0} u(x)\,df(x).$$

There is however an essential difference in the characters of the two integrals appearing in (7). We may verify directly that

$$(7') \quad \begin{aligned} \int_{a+0}^{b-0} f(x)\,du(x) &= \lim_{\varepsilon \doteq 0} \int_{a+\varepsilon}^{b-\varepsilon} f(x)\,du(x), \\ \int_{a+0}^{b-0} u(x)\,df(x) &= \lim_{\varepsilon \doteq 0} \int_{a+\varepsilon}^{b-\varepsilon} u(x)\,df(x), \end{aligned}$$

but if we complete the definitions of $u(x)$ and $f(x)$ in the closed interval (a, b) by defining $u(a)$ as $u(a+0)$ and $u(b)$ as $u(b-0)$ (in order to keep $u(x)$ continuous), and by defining $f(a)$ and $f(b)$ arbitrarily, we shall have

$$(8) \qquad \int_a^b f(x)\,du(x) = \int_{a+0}^{b-0} f(x)\,du(x)$$

but in general

$$\int_a^b u(x)\,df(x) \neq \int_{a+0}^{b-0} u(x)\,df(x).$$

4. Note on the second law of the mean. *If $f(x)$ is monotonic in the closed interval (a, b) and $u(x)$ continuous, there is a value ξ, $a \leqq \xi \leqq b$, and a value ξ', $a < \xi' < b$, such that*

$$
(9)\quad \begin{aligned}
\int_a^b f(x)\, u(x)\, d(x) &= f(a) \int_a^{\xi} u(x)\, dx + f(b) \int_{\xi}^b u(x)\, dx \\
&= f(a+0) \int_a^{\xi'} u(x)\, dx + f(b-0) \int_{\xi'}^b u(x)\, dx.
\end{aligned}
$$

Let $U(x)$ be an indefinite integral of $u(x)$. Then from (6) and (2) we have

$$
\begin{aligned}
\int_a^b f(x)\, dU(x) &= f(x)\, U(x)\Big]_a^b - \int_a^b U(x)\, df(x) \\
&= f(x)\, U(x)\Big]_a^b - U(\xi)\,[f(b) - f(a)] \\
&= f(a)\,[U(\xi) - U(a)] + f(b)\,[U(b) - U(\xi)].
\end{aligned}
$$

But as is seen directly from the Cauchy definition,

$$
\int_a^b f(x)\, dU(x) = \int_a^b f(x)\, u(x)\, dx.
$$

Hence

$$
\int_a^b f(x)\, u(x)\, dx = f(a) \int_a^{\xi} u(x)\, dx + f(b) \int_{\xi}^b u(x)\, dx.
$$

In a similar fashion, we obtain the other form, given above, by utilizing the open interval (a, b). That is,

$$
\begin{aligned}
\int_a^b f(x)\, dU(x) &= f(x)\, U(x)\Big]_{a+0}^{b-0} - \int_{a+0}^{b-0} U(x)\, df(x) \\
&= f(x)\, U(x)\Big]_{a+0}^{b-0} - U(\xi')\,[f(b-0) - f(a+0)], \quad (a < \xi' < b), \\
&= f(a+0)\,[U(\xi) - U(a+0)] + f(b-0)\,[U(b-0) - U(\xi')],
\end{aligned}
$$

from which the desired conclusion follows, since $U(a+0) = U(a)$, $U(b-0) = U(b)$.

5. Classical theorems on integrals and limits of integrals. A sequence of functions $f_n(x)$ is said to be bounded in their set, if there is a constant N, independent of n, such that $|f_n(x)| \leqq N$, in the interval in question. If we

have a sequence of functions $f_n(x)$ integrable over an interval (a, b) and bounded in their set, such that $\lim_{n=\infty} f_n(x) = f(x)$, we know that

$$\lim_{n=\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx,$$

provided that $f(x)$ is also integrable. This is Osgood's theorem. With integration in the Riemann sense we do not know that the limit function $f(x)$ is integrable. But the definition of integrability has been so extended by Borel and Lebesgue that if $f_n(x)$ is bounded and integrable, and $\lim_{n=\infty} f_n(x) = f(x)$, then $f(x)$, which of course is bounded, is also integrable. This theory has become classical.

If the $f_n(x)$ are not bounded in their set the corresponding theorems are more difficult to state. But the Lebesgue theory in the first place extends the definition of integration to functions which are not bounded, by defining the integral of such a function, if it is nowhere negative, as itself a limit; namely, the limit of the integral of the function

$$\begin{aligned} f_n(x) &= f(x) \text{ when } f(x) < n \\ &= n \qquad \text{when } f(x) \geqq n \end{aligned}$$

as n becomes infinite, if this limit exists. The function $f(x)$ is then said to be *summable* in (a, b). If $f(x)$ is not $\geqq 0$, it is said to be summable if the two functions $\frac{1}{2}\{|f(x)|+f(x)\}$ and $\frac{1}{2}\{|f(x)|-f(x)\}$ are summable, and the integral is defined as the difference of these two integrals. Sequences of summable functions are then dealt with by means of sufficient conditions; thus if $\lim_{n=\infty} f_n(x) = f(x)$, where the $f_n(x)$ are summable, a sufficient condition that $f(x)$ be summable and that its integral be the limit of the integral of $f_n(x)$ is that $|f_n(x)| \leqq \psi(x)$ were $\psi(x)$ is some summable function.

It is useful, however to have some form of necessary and sufficient condition. This may be obtained in terms of the concept of *absolute continuity*; a function $F(x)$ is *absolutely*

continuous if when we take a finite number of non-overlapping intervals (x_i', x_i''), $a \leqq x_i' < x_i'' \leqq b$, of total length $\leqq \delta$, we have

$$\sum_i |F(x_i'') - F(x_i')| \leqq m(\delta),$$

where $m(\delta)$ is a function of δ alone, which approaches zero with δ.

The function $m(\delta)$ is obviously unchanged if we take a denumerable infinity, instead of a finite number, of intervals of total length $\leqq \delta$. For if there were such a collection of intervals for which the given sum were $> m(\delta)$, there would be a finite number of them for which that sum would be greater than $m(\delta)$.

The absolute continuity of a set of such functions $F_n(x)$ is *uniform* if the function $m(\delta)$ is independent of n.

These concepts enable us to state what amounts to de la Vallée Poussin's theorem, of which those already given are special cases.

THEOREM OF DE LA VALLÉE POUSSIN. *Let $f_n(x)$ constitute a denumerable sequence of summable functions in an interval (a, b), not negative, with $\lim\limits_{n=\infty} f_n(x) = f(x)$. A necessary and sufficient condition that $f(x)$ shall be summable and that we shall have*

$$\lim_{n=\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

is that the absolute continuity of $F_n(x) = \int f_n(x)\,dx$ shall be uniform. If the $f_n(x)$ are of variable sign in (a, b), or if the sequence is non-denumerable, the condition is still sufficient.

A set of points on a line (or on a measurable arc) is said to be *of zero measure* if it can be contained in a denumerable set of intervals (or arcs) whose total length is arbitrarily small. A property is said to hold *almost everywhere* if it holds for all the points of a given segment except at most those which constitute some set of zero measure. A function is said to be *bounded almost everywhere,* or is still said to be *bounded,* if it may be defined or re-defined on a set of

zero measure so as to become bounded in the strict sense. To change the value of a function on a set of zero measure does not change the value of its Lebesgue integral, if it has one. A function is still said to be *summable* if it may be defined or re-defined on a set of zero measure so as to become summable.

It may be shown that a function of limited variation has a derivative almost everywhere, in spite of its possible discontinuities, and that this derivative is summable. An absolutely continuous function, which we see from the definition to be merely a special kind of continuous function of limited variation, is an indefinite integral of its derivative; conversely, the indefinite integral of a summable function is absolutely continuous. The total variation of an absolutely continuous function is the integral of the absolute value of its derivative.

The theorem of de la Vallée Poussin remains true if $\lim_{n=\infty} f_n(x) = f(x)$ almost everywhere, instead of everywhere.

6. The limits of Stieltjes integrals. THEOREM 2. *If $u_k(x)$ approaches $u(x)$ uniformly, both functions being continuous in (a, b), and $f(x)$ is of limited variation, then*

$$\lim_{k=\infty} \int_a^b u_k(x)\,df(x) = \int_a^b u(x)\,df(x).$$

In fact the absolute value of the difference between the integral for $u(x)$ and that for $u_k(x)$ is not greater than ηT, where $\eta \geqq |u-u_k|$ and approaches 0 as k becomes infinite. Similarly

$$\lim_{k=\infty} \int_{a+0}^{b-0} u_k(x)\,df(x) = \int_{a+0}^{b-0} u(x)\,df(x).$$

We say that $f_m(x)$ is of uniformly limited variation in (a, b), closed or open, if the total variation T_k is bounded for all k. This concept enables us to give a theorem proved independently by Helly and Bray, which we shall call the Helly-Bray theorem.*

* This theorem is discussed by Hildebrandt [Bulletin of the American Mathematical Society, vol. 28 (1922), p. 54] who gives a condition,

THEOREM 3. *Let (a, b) be a closed interval, $u(x)$ a continuous function in the interval, and E a set of points dense in the interval and including a and b. In the same interval let $f(x)$ be of limited variation and $f_m(x)$ a sequence of functions of uniformly limited variation, such that* $\lim_{m=\infty} f_m(x) = f(x)$ *if x is in E. Then*

$$\lim_{m=\infty} \int_a^b u(x)\, df_m(x) = \int_a^b u(x)\, df(x).$$

To prove this theorem, we notice that by taking δ small enough, and successive points of subdivision $x_1, x_2 \cdots, x_{n-1}$ close enough, in E, so that $x_{i+1} - x_i < \delta$, we can make

$$\left| \int_a^b u(x)\, df_m(x) - \sum_{i=0}^{n-1} u(x_i')\{f_m(x_{i+1}) - f_m(x_i)\} \right| < \varepsilon/4,$$

$$\left| \int_a^b u(x)\, df(x) - \sum_{i=0}^{n-1} u(x_i')\{f(x_{i+1}) - f(x_i)\} \right| < \varepsilon/4,$$

where ε is given arbitrarily, on account of the inequality (4). But also, $x, \cdots, x_{n-1}$ now being fixed and finite in number, by taking $m > m_0$, large enough, we can make

$$\left| \sum u(x_i')\{(f_m(x_{i+1}) - f_m(x_i)) - (f(x_{i+1}) - f(x_i))\} \right| < \varepsilon/2,$$

from our hypothesis about $\lim_{m=\infty} f_m(x)$ on E. From this the conclusion follows.

COROLLARY 1. *If in addition to the above hypotheses we have*

$$\lim_{m=\infty} f_m(a+0) = f(a+0), \quad \lim_{m=\infty} f_m(b-0) = f(b-0),$$

then also

$$\lim_{m=\infty} \int_{a+0}^{b-0} u(x)\, df_m(x) = \int_{a+0}^{b-0} u(x)\, df(x).$$

necessary as well as sufficient, for the limit of the integral to be the integral of the limit. Helly's formulation of the theorem [Wiener Sitzungsberichte, vol. 121 (IIa) (1912), p. 288] assumes $\lim_{m=\infty} f_m(x) = f(x)$ for all x. Bray's theorem [Annals of Mathematics, vol. 20 (1919), p. 180] is more easily applicable.

Let us note however that we can not deduce the hypothesis of this corollary merely from the properties of the set E.

COROLLARY 2. *If the conditions of both theorems on limits are assumed, we have*

$$\lim_{\substack{k=\infty \\ m=\infty}} \int_a^b u_k(x)\, d f_m(x) = \int_a^b u(x)\, d f(x),$$

without regard to the order in which k and m become infinite.

The theorems on the limit of a Stieltjes integral have a special interpretation if the interval (a, b) comprises the whole of a closed curve. For simplicity, let the curve be a circle, although we may use any simple closed rectifiable curve as well.

COROLLARY 3. *Let E denote a set of points dense on the circumference, and x the central angle or a continuous monotonic function of the central angle. Let $f(x)$, $f_m(x)$ denote functions of uniformly limited variation such that $\lim_{m=\infty} f_m(x) = f(x)$ if x is in E, and $u(x)$, $u_k(x)$ continuous functions on the circumference, such that $\lim_{k=\infty} u_k(x) = u(x)$ uniformly. Then for integration over the whole circumference, we have*

$$\int u(x)\, d f(x) = \lim_{\substack{k=\infty \\ m=\infty}} \int u_k(x)\, d f_m(x),$$

no matter how k and m become infinite.

7. Note on Lebesgue Integrals. Let now $u(x)$ be continuous, and $f(x)$ absolutely continuous, so that $f(x) = \int_a^x f'(x)\, dx + f(a)$. Then

$$\int_a^b u(x)\, f'(x)\, dx = \lim_{\delta=0} \sum_{i=0}^{n-1} u(x_i') \int_{x_i}^{x_{i+1}} f'(x)\, dx,$$

from the limit theorem on the Lebesgue integral; in fact, $u(x)$ is bounded. But also, from (5)

$$\lim_{\delta=0} \sum_{i=0}^{n-1} u(x_i') \int_{x_i}^{x_{i+1}} f'(x)\, dx = \int_a^b u(x)\, d f(x).$$

Hence

$$\int_a^b u(x)\,df(x) = \int_a^b u(x)f'(x)\,dx. \tag{10}$$

Also if $f(x)$ is of limited variation and $U(x)$ is absolutely continuous so that $U(x) = \int_a^x u(x)\,dx + U(a)$, where $u(x) = U'(x)$ almost everywhere and is summable, we have

$$\int_a^b f(x)\,u(x)\,dx = \int_a^b f(x)\,dU(x). \tag{11}$$

In fact, it is sufficient to consider a non-decreasing function $U(x)$ and write, by the law of the mean,

$$\int_a^b f(x)\,dU(x) = \lim_{\varepsilon=0}\sum_{i=0}^{n-1} f(x_i')\int_{x_i}^{x_{i+1}} u(x)\,dx.$$

But the function $f_i(x)$ defined as follows

$$f_i(x) = f(x_i'),$$

for $x_i \leqq x < x_{i+1}$, when $i+1 < n$, and for $x_{n-1} \leqq x \leqq x_n = b$ when $i+1 = n$, remains bounded, and has as a limit the function $f(x)$ on a set of points which includes those where $f(x)$ is continuous, that is, almost everywhere in (a, b). Hence

$$\sum_{i=0}^{n-1} f(x_i')\int_{x_i}^{x_{i+1}} u(x)\,dx = \int_a^b f_i(x)u(x)\,dx$$

and

$$\lim_{\varepsilon=0}\int_a^b f_i(x)u(x)\,dx = \int_a^b f(x)u(x)\,dx,$$

whence the conclusion follows.

By applying the idendity (11) to a monotonic function $f(x)$ it may be used in the proof of the second law of the mean. Hence we have the

Corollary to the Second Law of the Mean. *The second law of the mean holds if $f(x)$ is monotonic and $u(x)$ is summable in the Lebesgue sense.*

8. Convergence of Fourier series. We turn now to the subject of Fourier series, which is the second main subject of this chapter. If we write

$$(12) \qquad S_m(x) = \frac{a_0}{2} + \sum_{1}^{m}{}_k (a_k \cos kx + b_k \sin kx)$$

with a_k, b_k as the familiar Fourier coefficients

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\cos k\alpha \, d\alpha, \qquad b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\alpha)\sin k\alpha \, d\alpha$$

the function $f(x)$ being periodic with period 2π, and integrable in the ordinary or summable in the more general Lebesgue sense, an elementary manipulation of the trigonometric functions involved yields us the well known form

$$(13) \qquad S_m(x) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x+\alpha)\frac{\sin(m+\frac{1}{2})\alpha}{2\sin\frac{1}{2}\alpha}\, d\alpha.$$

And since $S_m(x) = a_0/2 = 1$ if $f(x) = 1$, we have the identity

$$1 = \frac{1}{\pi}\int_{-\pi}^{\pi}\frac{\sin(m+\frac{1}{2})\alpha}{2\sin\frac{1}{2}\alpha}\, d\alpha.$$

Consequently

$$S_m(x) - S(x) = (1/\pi)\int_{\pi}^{\pi}[f(x+\alpha) - S(x)]\frac{\sin(m+\frac{1}{2})\alpha\, d\alpha}{2\sin\frac{1}{2}\alpha},$$

where $S(x)$ is any given function of x. The question of the convergence of Fourier series is the question of the limit of this integral as m becomes infinite.

If small quantities approaching 0 uniformly for all x as m becomes infinite are discarded, it is easily shown that

$$(14) \quad \lim_{m=\infty} S_m(x) - S(x) = \lim_{m \quad \infty} (1/\pi)\int_0^{\varepsilon}\varphi(\alpha)\frac{\sin(m+\frac{1}{2})\alpha}{\alpha}\, d\alpha,$$

in which

$$\varphi(\alpha) = f(x+\alpha) + f(x-\alpha) - 2S(x),$$

and ε is any positive value.

In particular if $f(x)$ is continuous at some value x_0 and possesses there a derivative, we may assign $S(x_0)$ the value $f(x_0)$. In fact, with this definition, ε may be taken small enough so that for $0 < \alpha < \varepsilon$ the quantity $\varphi(\alpha)/\alpha$ is bounded $< K$ and

$$\lim_{m=\infty} S_m(x_0) - S(x_0) \leq (K/\pi)\lim_{m=\infty}\int_0^{\varepsilon}\sin\left(m+\frac{1}{2}\right)\alpha\, d\alpha = 0,$$

so that

$$\lim_{m=\infty} S_m(x_0) = S(x_0) = f(x_0).$$

In other words, the Fourier series converges to the value $f(x)$ at the point x_0. We are however more specially interested in *functions of limited variation.*

Since a function of limited variation possesses a derivative almost everywhere and has only a denumerable infinity of points of discontinuity, it follows at once that the Fourier series for such a function converges to the value of the function almost everywhere. A more complete result may however be obtained, and indeed, by elementary methods.

Let then $f(x)$ be a function of limited variation, and define

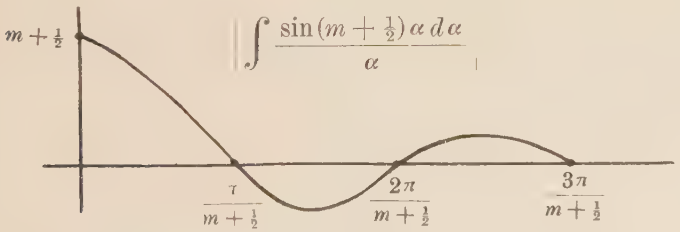
$$S(x) = \frac{1}{2}\{f(x+0)+f(x-0)\}.$$

The quantity $\varphi(\alpha)$ will also in this case for any x be a function of limited variation, as a function of α. Moreover it is continuous at $\alpha = 0$ and has there the value 0; and its positive and negative variation functions have the same properties.

Suppose first that $\varphi(\alpha)$ is monotonic, non-decreasing. We may then apply the second theorem of the mean, and write

$$\frac{1}{\pi}\int_0^{\varepsilon} \varphi(\alpha)\,\frac{\sin(m+\frac{1}{2})\alpha\, d\alpha}{\alpha} = \frac{\varphi(\varepsilon)}{\pi}\int_{\varepsilon'}^{\varepsilon} \frac{\sin(m+\frac{1}{2})\alpha\, d\alpha}{\alpha},$$

with $0 < \varepsilon' < \varepsilon$. But by reference to the accompanying figure it is seen that the greatest value that



can take on for any finite interval of integration is

$$\int_0^{\pi/(m+1/2)} \frac{\sin(m+\frac{1}{2})\alpha\, d\alpha}{\alpha} = \int_0^{\pi} \frac{\sin\alpha}{\alpha}\, d\alpha = C < \pi$$

Hence the integral above considered is $\leqq C\varphi(\varepsilon)/\pi$, which can be made as small as we please by taking ε small enough. In other words $S_m(x)-S(x)$ can be made as small as we please, by taking $m \geqq m_0$, large enough. The same conclusion holds if $\varphi(a)$ is not monotonic, since the contributions to $S_m(x)-S(x)$ by the positive and negative variations of $\varphi(a)$ taken separately may be made as small as we please. Hence, in general

$$\lim_{m=\infty} S_m(x)-S(x) = 0.$$

We have thus proved the following classical theorem.

THEOREM 4. *If $f(x)$ is a function of limited variation, periodic, with period 2π, its Fourier series converges everywhere and to the value*

$$\frac{1}{2}\{f(x+0)+f(x-0)\}.$$

The second theorem of the mean gives also a measure of the law of decrease of the Fourier coefficients for a function of limited variation. In fact, if $f(x)$ is monotonic, $-\pi<x<\pi$, non-decreasing, we have

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos kx\, dx$$

$$= \frac{1}{\pi} f(-\pi+0)\int_{-\pi}^{\xi'}\cos kx\, dx + \frac{1}{\pi} f(+\pi-0)\int_{\xi'}^{\pi}\cos kx\, dx$$

$$= (1/\pi)\{f(\pi-0)-f(-\pi+0)\}\frac{\sin k\xi'}{k}, \qquad -\pi<\xi'<\pi,$$

and

$$|a_k| \leqq \frac{1}{k\pi}\{f(\pi-0)-f(-\pi+0)\}.$$

Similarly

$$b_k = \frac{1}{\pi} f(-\pi+0)\left[-\frac{\cos kx}{k}\right]_{-\pi}^{\xi''}$$

$$+\frac{1}{\pi} f(\pi-0)\left[-\frac{\cos kx}{k}\right]_{\xi''}^{\pi}, \qquad -\pi<\xi''<\pi,$$

whence

$$|b_h| \leqq \frac{2}{k\pi}\{f(\pi-0)-f(-\pi+0)\}.$$

If $f(x)$ is not necessarily monotonic, the same relations hold separately for the Fourier coefficients a'_k, b'_k, a''_k, b''_k of the positive and negative variations of $f(x)$; and since a_k is $a'_k + a''_k$ and b_k is $b'_k + b''_k$, if we denote by T' the total variation of $f(x)$ in the open interval $-\pi < x < \pi$ we have

$$|a_k| \leqq \frac{T'}{k\pi}, \qquad |b_k| \leqq \frac{2\,T'}{k\pi}.$$

If T is the total variation in the corresponding closed interval, it may be shown that also $b_k \leqq T/k\pi$.

In fact

$$\begin{aligned} b_k &= -\frac{1}{\pi k}\int_{-\pi}^{\pi} f(x)\, d\cos kx \\ &= -\frac{1}{\pi k}\left[f(x)\cos kx\right]_{-\pi}^{\pi} + \frac{1}{\pi k}\int_{-\pi}^{\pi} \cos kx\, df(x), \end{aligned}$$

by an integration by parts of the Stieltjes integral. But the first term of this expression is zero, $f(x)$ being periodic in the closed interval. Moreover

$$\left|\int_{-\pi}^{\pi} \cos kx\, df(x)\right| \leqq \int_{-\pi}^{\pi} |\cos kx|\, dt(x) \leqq T,$$

which proves the desired inequality.*

We have thus established the following corollary for the functions concerned in Theorem 4.

Corollary. Let $f(x)$ be a function of limited variation, periodic with period 2π, and let T and T' be the total variations of $f(x)$ in any closed and any open interval, respectively, of length 2π. Then the following inequalities hold for the Fourier coefficients a_k, b_k

$$(15) \qquad |a_k| \leqq \frac{T'}{k\pi}, \qquad |b_k| \leqq \frac{T}{k\pi}, \qquad |b_k| \leqq \frac{2\,T'}{k\pi}.$$

Exercise. The reader may show by examples that these are the best inequalities obtainable in terms of the total variation.

* This short proof was given by Professor D. A. F. Robinson in my course at the University of Chicago, 1925.

A kind of converse problem to that of Fourier series is that of deciding when a given trigonometric series is convergent, and is a Fourier series for some function. Lebesgue's idea of measure has been a fundamental concept here also; and the problem has been extensively treated in a series of memoirs from Fatou (Acta Mathematica 1907) to Rademacher (Mathematische Annalen, 1922). For *normal functions* in general, the following theorem has been established.

THEOREM 5. *Given the series in terms of normal functions*

$$S = \sum_1^\infty c_i\, \varphi_i(x),$$

let $\sum c_i^2 (\log i)^2$ *converge. Then* S *converges to a function* $f(x)$ *which is summable which its square; and the* c_i *are the generalized Fourier coefficients for that function.*

9. Summability of series. We wish to discuss here merely one type of method of attributing a value to a divergent series, which is a natural extension of the concept of sum for a convergent series. This is a method of Borel.

If we have a series

$$u_0 + u_1 + u_2 + \cdots \tag{16}$$

we denote by S_n the sum

$$S_n = u_0 + u_1 + u_2 + \cdots + u_n \tag{17}$$

and if the series is convergent we define its value by the formula

$$S = \lim_{n=\infty} S_n.$$

In general, consider a sequence of values of r, $-r_1, r_2, \cdots$ with limit r', finite or infinite,—and let $a_k(r)$ be a sequence of not negative functions of r, non-increasing with k (i. e., such that $a_{k+1}(r) \leq a_k(r)$ for all r_i) and such that

$$\lim_{r_i = r'} a_k(r_i) = 1.$$

If the series

$$(18) \qquad \sigma(r) = \sum_1^{\infty} a_k(r)\, u_k$$

converges for all r_i and if $\lim_{i=\infty} \sigma(r_i)$ exists, we say that this limit is the sum, or sum in the extended sense, of the series (17). The following is the central theorem.

THEOREM 6. *If S exists, then $\sigma(r_i)$ is uniformly convergent for all r_i, and $S = \lim_{i\ \infty} \sigma(r_i)$.*

Let $R_n = u_n + u_{n+1} + \cdots$

$$\varrho_{n,m}(r) = a_n(r)\, u_n + a_{n+1}(r)\, u_{n+1} + \cdots + a_{n+m}(r)\, u_{n+m}.$$

Given ε we can choose N so that $|R_n| < \varepsilon$ for $n > N$, $m \geqq 0$. We have

$$\begin{aligned} \varrho_{n,m}(r) &= a_n(r)(R_n - R_{n+1}) + a_{n+1}(r)(R_{n+1} - R_{n+2}) + \cdots \\ &\quad \cdots + a_{n+m}(r)(R_{n+m} - R_{n+m+1}) \\ &= a_n R_n + (a_{n+1} - a_n) R_{n+1} + (a_{n+2} - a_{n+1}) R_{n+2} + \cdots \\ &\quad \cdots + (a_{n+m} - a_{n+m-1}) R_{n+m} - a_{n+m} R_{n+m+1} \end{aligned}$$

and

$$\begin{aligned} |\varrho_{n,m}(r)| &\leqq \varepsilon [a_n + (a_n - a_{n+1}) + (a_{n+1} - a_{n+2}) + \cdots \\ &\qquad \cdots + (a_{n+m-1} - a_{n+m}) + a_{n+m}] \\ &\leqq \varepsilon (2\, a_n + a_{n+m}) \\ &\leqq 3\, \varepsilon\, a_n(r). \end{aligned}$$

But $a_1(r_i)$ has a finite upper bound M, since $\lim_{i=\infty} a_1(r_i) = 1$; and $a_n(r_i) \leqq a_1(r_i)$. Hence $a_n(r_i) \leqq M$ and

$$|\varrho_{n,m}(r)| < 3\, M \varepsilon, \qquad \left.\begin{matrix} n \\ n+m \end{matrix}\right\} > N,$$

uniformly in r. In this range of uniformity is included the limit value $r = r'$. The series is therefore uniformly convergent for all values of r_i, including r', and the limit of the series as r_i tends to r' is S.

We note in passing that a similar theorem holds if $a_k(r_i)$ is non-decreasing, instead of non-increasing with k, and bounded, but the theorem is not useful.

We shall meet in the next chapter a type of this summation specially adapted to harmonic functions. One kind which has a peculiar interest in the case of trigonometric series is that of the arithmetic mean, applied by Fejér. In this case we write

$$(19) \qquad \sigma_n = \frac{S_0 + S_1 + \cdots S_{n-1}}{n} = \sum_{k=0}^{n-1} (1 - k/n)\, u_k$$

identical with the $\sigma(r_n)$ previously discussed if we let $r_n = n$, and

$$\begin{aligned} a_k(r_n) &= (1 - k/n) \qquad & k &= 0, 1, \cdots, n-1, \\ &= 0 & k &\geq n. \end{aligned}$$

Hence whenever the Fourier series converges, $\lim\limits_{n=\infty} \sigma_n$ converges and to the same value.

It may be readily calculated that

$$(19') \quad \sigma_m = \frac{1}{m\pi} \int_0^{\pi/2} \{f(x + 2\alpha) + f(x - 2\alpha)\} \left(\frac{\sin m\alpha}{\sin \alpha}\right)^2 d\alpha$$

from which it follows immediately that *if l and L are respectively the lower and upper bounds of the bounded function $f(x)$, then*

$$l \leq \sigma_m \leq L.$$

In fact (by putting $f(x) \equiv 1$) we have the identity

$$1 = \frac{2}{m\pi} \int_0^{\pi/2} \left(\frac{\sin m\alpha}{\sin \alpha}\right)^2 d\alpha.$$

From this follows Fejér's theorem:

THEOREM OF FEJÉR. *If $f(x)$ is bounded, by limits l and L, and if $|m a_m| \leq A$ and $|m b_m| \leq B$ for all m, a_m and b_m being the Fourier coefficients for $f(x)$, then any sum S_m of the Fourier series satisfies the inequality*

$$(20) \qquad l - (A + B) \leq S_m \leq L + A + B.$$

In fact, from (19),

$$\sigma_n = S_{n-1} - \frac{1}{n}\sum_0^{n-1} k u_k,$$

$$S_{n-1} = \sigma_n \quad + \frac{1}{n}\sum_0^{n-1} k u_k,$$

and the desired inequality follows. It applies in particular to all functions of limited variation.

It is worth while to notice another simple property of this method of summation.

THEOREM 7. *If $f(x)$, periodic, of period 2π, is of limited variation, then $\sigma_n(x)$ is of uniformly limited variation for all n.*

In fact, from

$$\sigma_m = \frac{1}{2m\pi}\int_{-\pi}^{\pi} f(x+\alpha)\left(\frac{\sin m\alpha/2}{\sin \alpha/2}\right)^2 d\alpha$$

which is equivalent to (19′), we have

$$\begin{aligned}\sum_1^n | \sigma_m(x_i') - \sigma_m(x_i'') | \\ &\leq \frac{1}{2m\pi}\int_{-\pi}^{\pi} \sum_1^n |f(x_i'+\alpha) - f(x_i''+\alpha)| \left(\frac{\sin m\alpha/2}{\sin \alpha/2}\right)^2 d\alpha. \\ &\leq T.\end{aligned}$$

CHAPTER II

FUNCTIONS HARMONIC WITHIN A CIRCLE

10. Preliminary theorems. We say that a function $u(M)$ is harmonic at a point M if it is continuous there with its first partial derivatives and has finite second derivatives with respect to x and y which satisfy Laplace's equation,*

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

If a function is harmonic at every point of a region, it is said to be harmonic in the region. There are some elementary facts about such functions which we may take for granted.

If a function is harmonic in a region Σ which contains the whole of a circle and its circumference, the value of the function at the center of the circle is the mean of its values on the circumference. Hence such a function cannot have a maximum or a minimum at an interior point of the region; for such a point could be made the center of a circle small enough to be entirely contained, with its circumference, in the region. It follows therefore that if we have a simple closed curve l, contained, with its interior, in the given region, there cannot be two functions, harmonic in the region, which are different at some point interior to l and have identical values on all the points of l; for their difference, which would be harmonic, would have a maximum or a minimum at an interior point of l. In particular l may be a circle.

If we make a transformation of the points of the plane which is conformal, directly or reversely, so that the region Σ is mapped univocally and continuously on another region Σ', a function harmonic in Σ is carried into a function harmonic

* In polar coördinates Laplace's equation is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0.$$

in Σ'. In general, an analytic function of a complex variable effects such a transformation. An example of a transformation which reverses the angles is furnished by inversion in a circle.

11. A preliminary result. Let us assume that our function $u(M)$ is harmonic inside a unit circle. The function $v(M)$ conjugate to $u(M)$ is then harmonic at every point inside the same circle; in particular it is single valued; and the function of a complex variable

$$\gamma(z) = u + iv, \qquad z = x + iy$$

is analytic inside the circle, and may be developed in a power series about the origin,

$$\gamma(z) = C_0 + C_1 z + C_2 z^2 + \cdots$$

convergent for $|z| = \sqrt{x^2+y^2} < 1$. If we write $C_k = A_k + i B_k$, and form the corresponding developments of the real and pure imaginary parts of $\gamma(z)$ they will also necessarily be convergent for $r = \sqrt{x^2+y^2} < 1$.

We write then

$$\begin{aligned} C_k z^k &= (A_k + i B_k) r^k (\cos k\theta + i \sin k\theta) \\ &= r^k \{A_k \cos k\theta - B_k \sin k\theta\} + i r^k \{B_k \cos k\theta + A_k \sin k\theta\}. \end{aligned}$$

Hence, for all points inside the unit circle,

$$u(M) = u(r,\theta) = \quad a_0/2 + \sum_1^\infty r^k(\quad a_k \cos k\theta + b_k \sin k\theta), \tag{2}$$

$$v(r,\theta) = -b_0/2 + \sum_1^\infty r^k(-b_k \cos k\theta + a_k \sin k\theta), \tag{2'}$$

where we have written $a_0/2 = A_0$, $b_0/2 = -B_0$; $a_k = A_k$, $b_k = -B_k$, for $k \geqq 1$. We note that

$$a_k r^k, \qquad b_k r^k$$

both tend to 0 as k becomes infinite if $r < 1$, whereas one of these expressions fails to remain finite if $r > 1$, if the

unit circle is the largest circle throughout the interior of which $u(M)$ is harmonic. That is the same as saying that the unit circle is the circle of convergence of $\gamma(z)$.

These inequalities show that the series for $u(M)$ is uniformly convergent for the points of any circle, of radius $r < 1$. Hence we may determine the coefficients a_k, b_k by multiplying through by $\cos kx$ and $\sin kx$ respectively and integrating from 0 to 2π. We find then that $r^k a_k$ and $r^k b_k$ are the Fourier coefficients of $u(M)$ considered as a function of θ on the circle of radius $r < 1$.

A second traditional method for the representation of a harmonic function is by means of Poisson's integral. In fact, if R is some fixed value, < 1, and $r < R$, we have

$$(3)\quad u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\varphi - \theta)} \cdot u(R, \varphi)\, d\varphi.$$

In order to prove this identity, we prove first the special cases of it where $u(r, \theta)$ is taken successively as 1, $r^n \cos n\theta$, and $r^n \sin n\theta$, functions which we know to be harmonic by direct differentiation or because they are the real parts of simple functions of a complex variable.

If z is the complex number

$$z = r\cos\theta + ir\sin\theta = re^{\theta i}$$

and, correspondingly, $t = Re^{\varphi i}$, and we let $\Re f(z)$ denote the real part of $f(z)$, we see by direct calculation that

$$\Re\left(\frac{t+z}{t-z}\right) = \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\varphi - \theta)}$$

so that, if we denote this fraction by $Q(\varphi)$, we have

$$\begin{aligned} Q(\varphi) &= \Re\left(\frac{1 + z/t}{1 - z/t}\right) = \Re\left[1 + \frac{2z}{t}\left(\frac{1}{1 - z/t}\right)\right] \\ &= 1 + \Re\left(\frac{2z}{t} + \cdots + \frac{2z^{p+1}}{t^{p+1}} + \cdots\right) \end{aligned}$$

since $|z| = r < |t| = R$. Hence

$$Q(\varphi) = 1 + 2\sum_{p=0}^{\infty}\left(\frac{r}{R}\right)^{p+1}\Re\{e^{(p+1)(\theta-\varphi)i}\}$$

$$= 1 + 2\sum_{p=0}^{\infty}\left(\frac{r}{R}\right)^{p+1}\{\cos(p+1)\varphi\cos(p+1)\theta + \sin(p+1)\varphi\sin(p+1)\theta\},$$

a series which for a given value of $r < R$ is uniformly convergent in θ and φ. Hence we may integrate term by term.

We have then the following identities

$$(4)\qquad \begin{aligned} \int_0^{2\pi} Q(\varphi)\,d\varphi &= 2\pi, \\ \int_0^{2\pi} Q(\varphi)\cos n\varphi\,d\varphi &= 2\pi\left(\frac{r}{R}\right)^n \cos n\theta, \\ \int_0^{2\pi} Q(\varphi)\sin n\varphi\,d\varphi &= 2\pi\left(\frac{r}{R}\right)^n \sin n\theta. \end{aligned}$$

remembering that

$$(4')\qquad \begin{aligned} \int_0^{2\pi} \sin n\varphi\cos p\varphi\,d\varphi &= 0, \\ \int_0^{2\pi} \sin n\varphi\sin p\varphi\,d\varphi &= \int_0^{2\pi}\cos n\varphi\cos p\varphi \\ &= \begin{cases} 0, & n \neq p, \\ \pi, & n = p. \end{cases} \end{aligned}$$

But equations (4) are merely special cases of the identity (3) which we wish to prove.

By means of (4) we have, for a finite sum,

$$(5)\qquad \begin{aligned} &\frac{a_0}{2} + \sum_1^n r^k(a_k\cos k\theta + b_k\sin k\theta) \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\varphi-\theta)} \\ &\qquad \times\left\{\frac{a_0}{2} + \sum_1^n R^k(a_k\cos k\varphi + b_k\sin k\varphi)\right\}d\varphi. \end{aligned}$$

From this we proceed to deduce first the following well-known theorem.

THEOREM 1. *Let $f(\varphi)$ be a function of limited variation with regular discontinuities, and periodic with period 2π. Then the function*

$$(6) \qquad u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta)} f(\varphi)\,d\varphi$$

is harmonic inside the circle of radius 1 and satisfies the condition

$$\lim_{r=1} u(r,\theta) = f(\theta)$$

for every value of θ. The same is true of the function $u(r,\theta)$ given by (2), when a_k and b_k are the Fourier coefficients for $f(\theta)$; and these two functions are the same.

That these functions are harmonic follows by direct differentiation, since $r<1$. Now the Fourier series for $f(\theta)$ converges everywhere, and to $f(\theta)$, by Theorem 4, Chap. I. But the representation given in (2) is merely a summation process, as described in Chap. I; hence it converges for $r<1$, and the limit as r tends to 1 is precisely $f(\theta)$. In fact, this is true for any sequence of values of r tending to 1.

Let us now apply the identity (5), taking $R=1$, to a finite number of terms of the expansion (2). We have then for the function $u(r,\theta)$ given by (2) the expression

$$u(r,\theta) = \lim_{n=\infty} \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta)} \times\left\{\frac{a_0}{2} + \sum_1^n (a_k\cos k\varphi + b_k \sin k\varphi)\right\} d\varphi,$$

where the quantity in brackets is the sum S_n of the first $2n+1$ terms of the Fourier series for $f(\varphi)$. But $f(\varphi)$ is of limited variation, and we may therefore apply Fejér's theorem, and write

$$|S_n| \leq |l| + |L| + \frac{2T}{\pi},$$

where the symbols have the significance already given them. Hence the sums S_n remain bounded in their set, and since they approach an integrable limit function, Osgood's theorem

tells us that the limit of the integral is the integral of the limit. But this last is precisely (6). In other words the function given by (2) is also given by (6), and the theorem is proved.

The function represented by (6) would be harmonic if $f(\varphi)$ were merely summable in the Lebesgue sense. But we do not yet know what values would be approached as r approached 1. And even in the case of Theorem 1 we have not discussed the question of the uniqueness of a harmonic function as determined by the boundary values $f(\varphi)$. But the preliminary result obtained in Theorem 1 will enable us to formulate necessary and sufficient conditions for such determinateness.

Let us return now to the proof of (3). According to (2), $r^k a_k$ and $r^k b_k$, and $R^k a_k$ and $R^k b_k$ are the Fourier coefficients of $u(r, \theta)$ and $u(R, \varphi)$ respectively. If we insert these values in (5), and take the limit as n becomes infinite, the identity (3) will be established, since $u(R, \varphi)$ is a function of φ of limited variation.

Exercise. Show from (3) that if $|u(R, \varphi)| < N$,

$$\left|\frac{\partial u}{\partial \theta}\right| < 2N\frac{r}{R}\,\frac{1+r/R}{(1-r/R)^3}, \qquad r<R.$$

Hence show that if $u(r, \theta)$ is harmonic in the entire plane and bounded it must reduce to a mere constant.*

12. Note on integral identities. A particular case of the equation so far considered is obtained by putting $f(\theta) = 1$ for all values of θ. In this case $u(r, \theta)$, as given by (1), where a_k and b_k are the Fourier constants for $f(\theta)$, reduces merely to unity, and the theorem just proved yields the identity

$$1 = \frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta)}\,d\varphi. \tag{7}$$

Let now (θ_1, θ_2), $\theta_2 > \theta_1$, be an arc of the circumference of length less than 2π, and apply the theorem to the function $f(\theta)$ which is defined as 1 within the arc and 0 outside the

* A closer inequality from the same source yields the quantity

$$(r/R)\,(4\,N/\pi)\,(1-r/R)^{-1}.$$

arc, the definition at the end points being arbitrary without affecting the value of the integral. We have then the identity

$$\lim_{r=1}\frac{1}{2\pi}\int_{\theta_1}^{\theta_2}\frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta)}\,d\theta$$

$$(8)\qquad = \begin{cases} 1\ , & \theta_1<\theta<\theta_2\,, \\ \frac{1}{2}\ , & \theta=\theta_1 \text{ or } \theta=\theta_2\,, \\ 0\ , & \theta \text{ outside } (\theta_1,\ \theta_2). \end{cases}$$

13. Digression. Functions of points and of point sets. If we imagine a distribution of positive and negative matter on the circumference of a circle, that distribution will be an additive function of point sets, and therefore the difference of two not negative additive functions of point sets. The essential character of the Poisson integral is that of being a sort of potential of a mass of this kind; in fact, the quantity

$$\frac{(1-r^2)\,m}{1+r^2-2r\cos(\varphi-\theta)}$$

represents the potential of what might be called a "circular doublet" of mass m; i. e., except for a constant it is the limit of the potential of two point masses made to increase in charge as they draw together, in such a way as to keep the potential at zero around the circle of radius 1. Hence the Poisson integral (3) represents the potential of circular doublets of strength density $f(\varphi)$. But it does not represent the potential of the most general distribution of such doublets.

EXERCISE. 1. Let B and B' be two points inverse with respect to the unit circle, and write $b=OB<1$, $b'=OB'>1$. At B place a mass of amount $-1/\log b=2/(b'-b)$ approx., and at B' an equal mass of opposite sign. Show, by means of similar triangles, that the potential of these two masses becomes 1 at a point on the circumference of the circle, taking the potential of mass 1 as $\log 1/\varrho$ at a distance ϱ from the mass. Find the limit of the potential due to these two masses as b approaches 1, at an arbitrary point $(r,\ \theta)$

interior to the circle. What is the value of this limit on the circumference?

2. At an arbitrary distance x on the axis OBB' place a mass of amount $\frac{1}{\log x}$. What is the limit of the potential due to this mass within the circle and on the circumference, as x is made to become infinite? The resulting potential may be called a level of strength -1.

3. Add the potential of problem 1 to a level of strength -1, and describe the result within and on the circumference of the circle.

Let then $\Phi(e)$ be an arbitrary additive function of point sets e on the circumference of radius 1; we can picture such a function $\Phi(e)$ as a total mass on the set e. The corresponding integral will be the Stieltjes integral

$$(9) \qquad u(r, \theta) = \frac{1}{2\pi} \int_C \frac{(1-r^2)\, d\Phi(e)}{1+r^2-2r\cos(\varphi-\theta)},$$

formed for the complete boundary of the circle, and defined as the limit of a sum in the same sort of way as the Stieltjes integral treated in Chap. I.

There is obviously only a denumerable infinity of points P_i, where the function of point sets will not vanish when the set is taken as a single point, i. e. $\Phi(P_i) \neq 0$. If we take the point a as distinct from such a point, we may define a function of θ, $F(\theta)$, almost everywhere by the equation

$$(9') \qquad F(\theta) = \Phi(a, \theta) + \text{const.}$$

where (a, θ) stands for the interval from a to θ. In fact this equation will yield a unique definition except at the points P_i; for these last points also we should get a unique definition if we specified the interval as open or closed. In this way, the value of $F(\theta)$ would be determined at every point of discontinuity as the value $F(\theta-0)$, or for every point of discontinuity as the value $F(\theta+0)$, respectively, according as we took the interval as open or closed. But it is better to leave the choice more arbitrary. We shall

limit ourselves, however, to such choices of the value of the function at a point of discontinuity as will render the sum total of all the numerical values of the jumps of the function bounded in magnitude,

$$\sum_i (\text{Oscill. at } P_i \text{ of } F(\theta)) < N.$$

In this way the function $F(\theta)$ will satisfy the definition of function of limited variation, given in the introduction. We may thus introduce extra points of discontinuity (which may be called unnecessary ones) if we desire.

We shall find that the most convenient choice for many purposes is to make the discontinuities regular, so that the equation

$$F(\theta) = \frac{1}{2}\{F(\theta+0)+F(\theta-0)\},$$

will be satisfied throughout. It will also, naturally, be satisfied for the positive, negative, and total variation functions of $F(\theta)$.

If we turn to the reciprocal problem, we find as a familiar theorem in the theory of functions of a real variable, that given a function $F(\theta)$ of limited variation, there is one and only one additive function of point sets $\Phi(e)$ which accords with it by the equation

$$(10) \qquad F(\theta_1)-F(\theta_2) = \Phi(\theta_1, \theta_2),$$

θ_1 and θ_2 being points of continuity for $F(\theta)$.*

There is a further point that is worthy of remark. Since our fundamental interval is the circumference of a circle, we may have to consider values of θ less than 0 and greater than 2π. For this reason we must assign the values of $F(\theta)$ outside this interval in accordance with the values which it takes on within it. Accordingly we write

* In fact, an additive function of point sets is completely determined by the values which it takes on the meshes of some net [de la Vallée Poussin, *Intégrales de Lebesgue*, Paris (1916), Chap. VI].

$$F(2\pi+\theta) = F(\theta) + \Phi(c) \tag{11}$$

where $\Phi(c)$ is the value of $\Phi(e)$ for the whole circumference, and by (9′) the value of $F(2\pi) - F(0)$.

Let us form now the integral

$$u(M) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)\, dF(\varphi)}{1+r^2-2r\cos(\varphi-\theta)}, \tag{12}$$

where $F(\varphi)$ corresponds to $\Phi(e)$ by (10) and (11). We see that (12) and (9) are identical. In fact, both may be compared with the same Riemann sum, formed on points φ_i which are points of continuity for $F(\varphi)$. We are about to get necessary and sufficient conditions for the integral (12), and thus for (9), and incidentally for (6).

14. Properties of the Poisson-Stieltjes integral. *We consider the fundamental formula, in terms of a Poisson-Stieltjes integral,*

$$u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)\, dF(\varphi)}{1+r^2-2r\cos(\varphi-\theta)}, \tag{A}$$

in which $F(\varphi)$ is of limited variation, such that $F(2\pi+\theta) = F(\theta)+F(2\pi)-F(0)$, and $r<1$. We proceed to deduce from (A) a number of significant properties of the function $u(r, \theta)$.

We have immediately

THEOREM 2. *The function given by* (A) *is harmonic inside the circle of radius* 1.

As a matter of fact we would be justified in proving this by differentiating the Stieltjes integral under the integral sign, since $r<1$. In order to avoid the unfamiliar process, however, let us integrate (A) by parts. We have

$$u(r, \theta) = \frac{1}{2\pi} \frac{F(\varphi)(1-r^2)}{1+r^2-2r\cos(\varphi-\theta)}$$
$$- \frac{1}{2\pi}\int_0^{2\pi} F(\varphi)\, d\frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta)},$$

or

$$
(13)\qquad
\begin{aligned}
u(r,\,\theta) = {} & \frac{F(2\pi)-F(0)}{2\pi}\,\frac{1-r^2}{1+r^2-2r\cos\theta} \\
& + \frac{1}{2\pi}\int_0^{2\pi} F(\varphi)\frac{2r(1-r^2)\sin(\varphi-\theta)}{(1+r^2-2r\cos(\varphi-\theta))^2}\,d\varphi .
\end{aligned}
$$

But (13) gives $u(r,\,\theta)$ in terms of the usual type of integral. We may then differentiate under the integral sign and substitute in Laplace's equation. Thus the fact will be verified.

In the expression just written, r being <1, we may also integrate under the integral sign with respect to θ. This gives us

$$
\begin{aligned}
\int_{\theta_1}^{\theta_2} u(r,\,\theta)\,d\theta = {} & \frac{F(2\pi)-F(0)}{2\pi}\int_{\theta_1}^{\theta_2}\frac{1-r^2}{1+r^2-2r\cos\theta}\cdot d\theta \\
& + \frac{1}{2\pi}\int_0^{2\pi} F(\varphi)\frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta_2)}\cdot d\varphi \\
& - \frac{1}{2\pi}\int_0^{2\pi} F(\varphi)\frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta_1)}\cdot d\varphi ,
\end{aligned}
$$

and if we denote by $F(r,\,\theta)$ the quantity

$$
F(r,\,\theta) = \int_0^{\theta} u(r,\,\theta)\,d\theta ,
$$

we may let r approach 1 in the above formula, and with reference to (8) of Art. 12, obtain for $\theta \neq 0$, $\theta \neq 2\pi$, the result

$$
\begin{aligned}
\lim_{r=1} F(r,\,\theta) = {} & \frac{F(2\pi)-F(0)}{2} + \frac{1}{2}\{F(\theta+0)+F(\theta-0)\} \\
& - \frac{1}{2}\{F(2\pi-0)+F(0+)\} \\
= {} & \frac{F(2\pi)-F(0)}{2} + \frac{1}{2}\{F(\theta+0)+F(\theta-0)\} \\
& - \frac{1}{2}\{F(2\pi)-F(0)+F(0-)+F(0+)\}
\end{aligned}
$$

or

$$
(14) \qquad \begin{aligned} \lim_{r=1} F(r, \theta) = \frac{1}{2}\{F(\theta+0)+F(\theta-0)\} \\ -\frac{1}{2}\{F(0+)+F(0-)\}, \end{aligned}
$$

a formula which may also be verified directly for $\theta = 0$ and $\theta = 2\pi$, reducing to 0 in the former case and to $F(2\pi) - F(0)$ in the latter. Hence we have the theorem:

THEOREM 3. *If* $u(r, \theta)$ *is given by* (A), *the function* $F(r, \theta) = \int_0^\theta u(r, \theta)\, d\theta$ *has a limit, as* r *approaches* 1, *for every value of* θ. *This limit is given by* (14).

It may be noticed that if the discontinuities of $F(\theta)$ are regular in an open interval which includes the closed interval $(0, 2\pi)$, (14) yields the formula

$$
(15) \qquad \lim_{r=1} F(r, \theta) = F(\theta) - F(0), \qquad 0 \leqq \theta \leqq 2\pi.
$$

We can also obtain important results about $u(r, \theta)$ itself. Since

$$
u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)\, dF_1(\varphi)}{1+r^2-2r\cos(\varphi-\theta)} - \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)\, dF_2(\varphi)}{1+r^2-2r\cos(\varphi-\theta)},
$$

where $F_1(\varphi)$ is the positive, and $-F_2(\varphi)$ the negative variation of $F(\varphi)$, we see that $u(r, \theta)$ is, within the circle, the difference of two not negative harmonic functions. For each of the integrals (as we see by the corresponding Riemann sum) yields an essentially not negative quantity. Hence:

THEOREM 4. *If* $u(r, \theta)$ *is given by* (A) *it is the difference of two not negative functions harmonic within the unit circle.*

If we denote by $T(\theta)$ the total variation function of $F(\theta)$, and by $U(r, \theta)$ the harmonic function

$$
U(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta)} \cdot dT(\varphi),
$$

then obviously

$$|u(r, \theta)| \leqq U(r, \theta),$$

and

$$\int_0^{2\pi} |u(r, \theta)|\, d\theta \leqq \int_0^{2\pi} U(r, \theta)\, d\theta.$$

But we know from the mean value property of harmonic functions given in Art. 10, that this latter quantity is a constant, independent of r, equal in fact to 2π times the value of U at the center of the circle. By (14), however, this value is also $T(2\pi) - T(0)$. We have then the following result.

Theorem 5. *The quantity* $\int_0^{2\pi} |u(r, \theta)|\, d\theta$ *is bounded for* $r < 1$, $\leqq T(2\pi) - T(0)$. *In other words the function* $F(r, \theta)$ *is of uniformly limited variation in* θ *for all* r.

In fact, the total variation of $F(r, \theta)$ in the closed interval 0 to 2π is equal precisely to $\int_0^{2\pi} |u(r, \theta)|\, d\theta$.

We can, as a further property, evaluate directly the limit as r approaches 1 of the integral $\int_{\theta_1}^{\theta_2} |u(r, \theta)|\, d\theta$. For this purpose let us write $\bar{F}(\theta) = (1/2)\,[F(\theta+0) + F(\theta-0)]$, and take $\bar{T}(\theta)$ as the total variation function of $\bar{F}(\theta)$. By introducing again the function $\bar{U}(r, \theta)$ corresponding to the $U(r, \theta)$ above, we have by (14) immediately

$$\lim_{r=1} \int_{\theta_1}^{\theta_2} |u(r, \theta)|\, d\theta \leqq \bar{T}(\theta_2) - \bar{T}(\theta_1), \qquad \theta_2 \geqq \theta_1 \geqq 0.$$

But also, we can show that

$$\lim_{r=1} \int_{\theta_1}^{\theta_2} |u(r, \theta)|\, d\theta > \bar{T}(\theta_2) - \bar{T}(\theta_1) - \varepsilon$$

no matter what $\varepsilon > 0$ is given, by taking r near enough to 1, i. e. $r > r_\varepsilon$. From this the theorem will follow.

In fact, we can choose a finite number of subintervals in (θ_1, θ_2), namely (θ_j', θ_j''), $j = 1, 2, \cdots, n$, so that

$$T(\theta_2) - T(\theta_1) - \sum_{1}^{n}{}_j\, |[F(\theta_j'') - F(\theta_j')]| < \varepsilon/2.$$

Now $\lim\limits_{r=1} F(r, \theta) = \bar{F}(\theta) - F(0)$, hence we can take r near enough to 1 so that for every one of this finite number $2n$ of values of θ_j', θ_j'', $|[\bar{F}(\theta) - F(0)] - F(r, \theta)| < \dfrac{\varepsilon}{4n}$. Therefore

$$\big|\,|F(\theta_j'') - F(\theta_j')| - |F(r, \theta_j'') - F(r, \theta_j')|\,\big| < \varepsilon/2n.$$

and finally, with the help of the previous inequality,

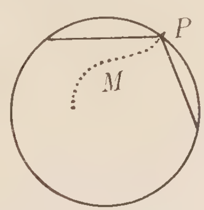
$$\int_{\theta_1}^{\theta_2} |u(r, \theta)|\, d\theta > \sum_{1}^{n}{}_j |F(r, \theta_j'') - F(r, \theta_j')| > \bar{T}(\theta_2) - \bar{T}(\theta_1) - \varepsilon.$$

Hence $\lim\limits_{r-1} \displaystyle\int_{\theta_1}^{\theta_2} |u(r, \theta)|\, d\theta > \bar{T}(\theta_2) - \bar{T}(\theta_1) - \varepsilon$, whatever ε. Hence we have the theorem:

THEOREM 6. *Let $T(\theta)$ be the total variation function of $F(\theta)$ after its discontinuities have been made regular. Then*

$$\lim_{r=1} \int_{\theta_1}^{\theta_2} |u(r, \theta)|\, d\theta = T(\theta_2) - \bar{T}(\theta_1).$$

15. Continuation of the preceding. Behaviour of $u(r, \theta)$ in the neighborhood of the boundary. We assume that $u(r, \theta)$ is still given by (A) with $F(2\pi + \theta) = F(\theta) + F(2\pi) - F(0)$ and proceed to establish the following result.



THEOREM 7. *Let P be a point on the circumference where $F(\varphi)$ is continuous and has a unique derivative $F'(\varphi) = f$. We draw any two chords of the circle at P. If M approaches P in such a way as to remain between these two chords, then* $\lim\limits_{M \to P} u(M) = f$.

We shall say that M approaches P *in the wide sense,* if it approaches P as in Theorem 7, and that $u(M)$ takes on the values $f(P)$ *in the wide sense.*

Since $F(\varphi)$ is continuous and possesses a unique derivative *almost everywhere,* this result applies almost everywhere on the circumference. Hence the proposition:

COROLLARY. *The harmonic function given by (A) takes on the boundary values given by $F'(\varphi) = f(\varphi)$ in the wide sense at almost all the points of the boundary.*

In order to prove the theorem we shall take the point P at the position (1,0) and by making use of the relation $F(2\pi + \theta) = F(\theta) + \text{const.}$, write the integral in the form (16)

$$u(r, \theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r^2)\, dF(\varphi)}{1 + r^2 - 2r\cos(\varphi - \theta)}. \tag{16}$$

Consider a sequence of points $(r, \theta_r) = P_r$ such that $\lim_{r\to 1} \theta_r = 0$. The hypothesis of the theorem, relative to the manner of approach, is equivalent to saying that there is some finite N such that $|\theta_r|/(1-r) < N$.

We may split the integral into three, corresponding to the equation

$$F(\varphi) = F(0) + \varphi f(0) + \varphi \eta(\varphi),$$

where $f(0) = F'(0)$ and $\varphi\eta(\varphi)$ is a function of limited variation such that $\lim_{\varphi = 0} \eta(\varphi) = 0$. In fact

$$\eta(\varphi) = \frac{F(\varphi) - F(0)}{\varphi} - f(0).$$

We have, therefore the relation

$$\begin{aligned} u(P_r) &= \frac{f(0)}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 + r^2 - 2r\cos(\varphi - \theta_r)} \cdot d\varphi \\ &+ \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{(1 - r^2)\, d(\varphi\eta(\varphi))}{1 + r^2 - 2r\cos(\varphi - \theta_r)}, \end{aligned} \tag{17}$$

in which the first term of the right hand member is identically equal to $f(0)$ for all values of $r < 1$.

We wish to show that the second term of (17) has a limiting value ξ as r tends to 1 such that $|\xi| \leq \varepsilon$, no matter what ε has been assigned. For this purpose, define η_α as the upper bound of $\eta(\varphi)$ in the interval $|\varphi| \leq \alpha$, and fix α as a quantity small enough so that

$$\eta_\alpha \leqq \frac{\varepsilon}{\frac{2N}{\pi}+1}$$

which is of course possible in accordance with our hypothesis. We have, if the limit exists,

$$\xi = \lim_{r=1} \frac{1}{2\pi} \int_{-\alpha}^{\alpha} \frac{(1-r^2)\, d(\varphi\eta(\varphi))}{1+r^2-2r\cos(\varphi-\theta_r)},$$

since ultimately $|\theta_r| < \alpha$, say $\leqq \alpha/2$.

If we denote the integral in the above expression by I_r, and perform an integration by parts, we have

$$I_r = \left[\frac{(1-r^2)\varphi\eta(\varphi)}{1+r^2-2r\cos(\varphi-\theta_r)}\right]_{-\alpha}^{\alpha} + \int_{-\alpha}^{\alpha} \frac{2r(1-r^2)\sin(\varphi-\theta_r)}{[1+r^2-2r\cos(\varphi-\theta_r)]^2}\, \varphi\eta(\varphi)\, d\varphi,$$

of which the first term has the limit 0 as r approaches 1. Consider now the integral which is left, and denote it by J_r. We have

$$J_r = \theta_r \int_{-\alpha}^{\alpha} \frac{2r(1-r^2)\sin(\varphi-\theta_r)\eta(\varphi)}{[1+r^2-2r\cos(\varphi-\theta_r)]^2}\, d\varphi + \int_{-\alpha}^{\alpha} \frac{2r(1-r^2)\sin(\varphi-\theta_r)(\varphi-\theta_r)\eta(\varphi)}{[1+r^2-2r\cos(\varphi-\theta_r)]^2} \cdot d\varphi.$$

The second integral, which we may call I_r'', has an integrand which is constantly of one sign except for the factor $\eta(\varphi)$. Hence, by the law of the mean,

$$|I_r''| \leqq \eta_\alpha \int_{-\alpha}^{\alpha} \frac{2r(1-r^2)(\varphi-\theta_r)\sin(\varphi-\theta_r)\, d\varphi}{[1+r^2-2r\cos(\varphi-\theta_r)]^2}$$

$$\leqq \eta_\alpha \left[(\varphi-\theta_r)\frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta_r)}\right]_{-\alpha}^{\alpha} + \eta_\alpha \int_{-\alpha}^{\alpha} \frac{(1-r^2)\, d\varphi}{1+r^2-2r\cos(\varphi-\theta_r)}.$$

The first term of this expression has the limit 0, and the second term is less than the value it would have if the integration were from $-\pi$ to π, that is, $< 2\pi\eta_\alpha$. Hence, if a limit exists,

$$\lim_{r=1} |I_r''| \leqq 2\pi\eta_\alpha.$$

With regard to the first term in the expression for J_r, which we shall call I_r', we have

$$I_r' < \eta_\alpha \, |\theta_r| \left\{ \int_{-\alpha}^{\theta_r} - \frac{2r(1-r^2)\sin(\varphi-\theta_r)\,d\varphi}{[1+r^2-2r\cos(\varphi-\theta_r)]^2} \right.$$
$$\left. + \int_{\theta_r}^{\alpha} \frac{2r(1-r^2)\sin(\varphi-\theta_r)\,d\varphi}{[1+r^2-2r\cos(\varphi-\theta_r)]^2} \right\}.$$

The first term of this is merely

$$\eta_\alpha |\theta_r| \left\{ \frac{1+r}{1-r} - \frac{1-r^2}{1+r^2-2r\cos(\alpha-\theta_r)} \right\},$$

of which the second part has the limit 0 and the first part is in numerical value $< 2N\eta_\alpha$, by hypothesis. The second term in the preceding inequality can be handled in the same way, and therefore, if a limit exists,

$$\lim_{r=1} |I_r'| \leqq 4\eta_\alpha N.$$

Consequently, if a limit exists,

$$\xi \leqq \frac{2\pi\eta_\alpha + 4N\eta_\alpha}{2\pi} = \epsilon$$

and therefore all these limits do exist, and

$$\xi = 0.$$

The inequalities given above make it possible to investigate the approach when the conditions of the hypothesis are not satisfied as to the manner of approach, and also when $F(\varphi)$ is discontinuous in certain ways at $\varphi = 0$. The result just given is a special case of Fatou's well known theorem. Fatou shows that if

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)\,2\,r\sin(\varphi-\theta)}{[1+r^2-2\,r\cos(\varphi-\theta)]^2}\,F(\varphi)\,d\varphi,$$

with $F(\varphi)$ summable in the Lebesgue sense, $\lim u(M) = F'(P)$ in the wide sense, wherever $F'(P)$ exists.* Theorem 7 may be obtained from this by means of an integration by parts.

EXERCISE. Extend the above theorem to the case where $F(\varphi)$ is bounded, continuous in the neighborhood of $(1, 0)$, with a unique derivative at $(1, 0)$, but not necessarily of bounded variation. For convenience take $F(\varphi)$ as continuous on the circle except for a finite number of discontinuities, and extend siutably the definition of the Stieltjes integral.

16. The Poisson integral: $F(\varphi)$ absolutely continuous. If $F(\varphi)$ is absolutely continuous, equal to a constant plus $\int_0^\varphi f(\varphi)\,d\varphi$, the Stieltjes integral reduces, as we have seen from Chapter I, to a Lebesgue integral, and we have

$$(B) \qquad u(r,\theta) = (1/2\pi)\int_0^{2\pi} \frac{(1-r^2)\,f(\varphi)\,d\varphi}{1+r^2-2\,r\cos(\varphi-\theta)}.$$

In (B), *the $f(\varphi)$ represents an arbitrary function summable in the Lebesgue sense,* and is consequently the unique derivative of $F(\varphi) = \text{const.} + \int_0^\varphi f(\varphi)\,d\varphi$ $\left(\text{or of } F(e) = \int_e f(\varphi)\,d\varphi\right)$ for almost all φ. It is worth while to restate the theorems already given for this case.

THEOREM 8. *If $u(r,\theta)$ is given by (B), it takes on the boundary values $f(\varphi)$ almost everywhere, in the wide sense; moreover its absolute value takes on the boundary values $|f(\varphi)|$ in the same sense; and the following equations are valid:*

$$(18) \qquad \lim_{r=1}\int_{\theta_1}^{\theta_2} u(r,\theta)\,d\theta = \int_{\theta_1}^{\theta_2} f(\theta)\,d\theta$$

$$(19) \qquad \lim_{r=1}\int_{\theta_1}^{\theta_2} |u(r,\theta)|\,d\theta = \int_{\theta_1}^{\theta_2} |f(\theta)|\,d\theta.$$

* Acta Mathematica, vol. 30 (1906), p. 345 and p. 357. Compare Kellogg's theorem on approach along curves with finite terminal curvature orthogonal to the boundary of a finitely connected region "subject to condition $(A^{(2)})$" [Transactions of the American Mathematical Society, vol. 13 (1912), p. 128].

There is a further property of the function $F(r, \theta) = \int_0^\theta u(r, \theta)\, d\theta$ which it will be shown later is characteristic for the equation (B), and which we now proceed to demonstrate.

THEOREM 9. *If $u(r, \theta)$ is given by (B), the absolute continuity of the function $F(r, \theta)$, is uniform for all values of r, $r < 1$.*

For the purpose merely of this theorem we shall define $u(1, \theta)$ to be $f(\theta)$. Since the uniform absolute continuity of the integral comes from the bounded character of the function when r is less than some constant r_1, r_1 being < 1, it is sufficient to consider r_1 in the closed interval $(r_1, 1)$. It is also sufficient to consider $f(\theta) \geqq 0$ and therefore $u(r, \theta) \geqq 0$, since $f(\theta)$ otherwise is merely the difference of two such non-negative functions. In fact if $f(\theta)$ is the difference $f_1(\theta) - f_2(\theta)$ of two non-negative functions, then the corresponding functions $u_1(r, \theta)$, $u_2(r, \theta)$ given by (B) are not negative, and

$$\int_e |u(r, \theta)|\, d\theta \leqq \int_e u_1(r, \theta)\, d\theta + \int_e u_2(r, \theta)\, d\theta,$$

where e denotes any set consisting of a finite number of non-overlapping intervals. Let us then assume $f(\theta) \geqq 0$.

Let r_0 be any value of r in the closed interval $(r_1, 1)$ and r_m any denumerable sequence of values of r which tend toward it as a limit. Then

$$(20) \qquad \lim_{m=\infty} \int_0^\theta u(r_m, \theta)\, d\theta = \int_0^\theta u(r_0, \theta)\, d\theta .$$

This property is obvious if $r_0 < 1$. If $r_0 = 1$ it is given by Theorem 8. From Theorem 8 we know also that $\lim_{m=\infty} u(r_m, \theta) = u(r_0, \theta)$ everywhere or almost everywhere. Hence the Theorem of de la Vallée Poussin tells us that the absolute continuity of $F(r_m, \theta)$ is uniform for all m.

Suppose now that the absolute continuity of $F(r, \theta)$ were not uniform for *all* r, $r_1 \leqq r \leqq 1$. Then we could find an ε such that no matter what $\delta_i > 0$ we chose there would be

a value of $r = r_i$ and a finite number of intervals e_i of total measure $< \delta_i$, such that

$$\int_{e_i} u(r_i, \theta)\, d\theta > \epsilon. \tag{21}$$

Hence if we chose $\delta_1, \delta_2, \delta_3, \cdots$ such that $\lim\limits_{i=\infty} \delta_i = 0$, we should have a sequence of values of r, namely $r_1, r_2, \cdots$ which would have at least one limiting point, say r', in the closed interval $(r_1, 1)$.* We could then choose a subsequence from them, say $r_{n_1}, r_{n_2}, \cdots$ which would approach r' as a true limit, and for which (21) would hold. But this is in contradiction with our previous result, taking

$$r_0 = r'.$$
$$r_m = r_{n_m}.$$

Hence the theorem is proved.

* The case where there are only a finite number of different values of r_i is trivial.

CHAPTER III

NECESSARY AND SUFFICIENT CONDITIONS
THE DIRICHLET PROBLEMS FOR THE CIRCLE

17. Fundamental theorem and lemma. The principal object of the present chapter is to prove a theorem which gives sufficient conditions for the Poisson-Stieltjes and Poisson integrals. The necessity of them has already been proven, so that these conditions are both *necessary and sufficient.*

THEOREM 1. FUNDAMENTAL THEOREM. *Let* $u(r, \theta)$ *be harmonic inside the unit circle, denote as before* $\int_0^\theta u(r, \theta)\, d\theta$ *by* $F(r, \theta)$, *and let* $r_1, r_2, \cdots$ *be a sequence of values of* r, $r < 1$, *such that* $\lim\limits_{n=\infty} r_n = 1$. *If now*

(i) $F(r_n, \theta)$ is of limited variation uniformly for all n *then* $u(r, \theta)$ *is given by a Poisson-Stieltjes integral* (A)

$$\text{(A)} \quad u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta)}\, dF(\varphi),$$

where $F(\varphi)$ *is of limited variation and* $F(2\pi+\varphi) = F(\varphi)+F(2\pi)-F(0)$; *and if*

(ii) $F(r_n, \theta)$ is absolutely continuous uniformly for all n *then* $u(r, \theta)$ *is given by a Poisson integral* (B)

$$\text{(B)} \quad u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta)} f(\varphi)\, d\varphi,$$

where $f(\varphi)$ *is summable in the Lebesgue sense.*

The condition (i) is equivalent to the condition

$$\text{(i)} \qquad \int_0^{2\pi} |u(r_n, \theta)|\, d\theta < N, \qquad \text{for all } n.$$

The condition (ii) can also be stated in the form:

(ii) Given ε, arbitrarily, we can find δ_ε such that if e is a set consisting of a finite number of intervals of total measure $< \delta_\varepsilon$, then

$$\int_e |u(r_n, \theta)|\, d\theta \leq \varepsilon, \qquad \text{for all } n, e.$$

The condition (ii) obviously implies the condition (i).

If the function is harmonic within the unit circle, it is given by (2) Chap. II

$$u(r, \theta) = \frac{a_0}{2} + \sum_1^\infty r^n (a_n \cos n\theta + b_n \sin n\theta).$$

Since a constant $a_0/2$ satisfies the conditions (i), (ii), and is moreover given by a Poisson integral

$$\frac{a_0}{2} = \frac{1}{2\pi} \int_0^{2\pi} \frac{a_0/2\,(1-r^2)}{1+r^2-2r\cos(\varphi-\theta)}\, d\varphi,$$

we may omit it from consideration, writing $u(r, \theta) = 0$ when $r = 0$ and

$$(1) \qquad u(r, \theta) = \sum_1^\infty r^n (a_n \cos n\theta + b_n \sin n\theta), \qquad r < 1.$$

The function $F(r, \theta)$ is then periodic as a function of θ with period 2π; in fact

$$\begin{aligned} F(r, \theta) &= \sum_1^\infty \frac{r^n}{n} (a_n \sin n\theta + (1 - b_n) \cos n\theta) \\ &= F_1(r, \theta) - F_1(r, 0) \end{aligned}$$

where

$$(2) \qquad \begin{cases} F_1(r, \theta) = \sum_1^\infty r^n (A_n \cos n\theta + B_n \sin n\theta), \\ n A_n = -b_n, \\ n B_n = a_n. \end{cases}$$

LEMMA. *The series for F_1 (r, θ) converges when $r = 1$ and represents a function of limited variation. It is the Fourier series for that function.*

In fact, $F_1(r, \theta)$ is of limited variation uniformly in r, since its total variation in θ is precisely the total variation of $F(r, \theta)$. Hence by the Corollary to Theorem 4, Chap. I,

$$r^n A_n \leqq T/\pi n, \qquad r^n B_n \leqq T/\pi n,$$

where T is independent of r, $r < 1$. Consequently we have

$$(3) \qquad A_n \leqq T/\pi n, \qquad B_n \leqq T/\pi n.$$

By Theorem 5, Chap. I, we know that the series

$$\sum_{1}^{\infty} A_n \cos n\theta + B_n \sin n\theta$$

converges almost everywhere in the interval $(0, 2\pi)$ and that it represents there a function $F_1(\theta)$ which with its square is summable over that interval; moreover that A_n, B_n are the Fourier coefficients for $F_1(\theta)$.

Now $F_1(\theta)$ is of limited variation on the set E where it is defined. The set E is dense on the circumference of the unit circle, for since the complementary set CE is of measure 0 there can be no arc without points of E. Hence by Corollary 2 of Theorem 1, Chap. I, $F_1(\theta)$ can be extended so as to be of limited variation over the closed interval $(0, 2\pi)$, periodic and with regular discontinuities; and since we are changing $F_1(\theta)$ only on a set of measure 0 we shall still have A_n, B_n as its Fourier coefficients. But now, since $F_1(\theta)$ is of limited variation, periodic and with regular discontinuities, its Fourier series converges everywhere to $F_1(\theta)$, and, moreover, everywhere

$$\lim_{r \to 1} F_1(r, \theta) = F_1(\theta), \tag{4}$$

from Theorem 6, Chap. I. This proves the lemma and also the following corollary.

Corollary 1. *The function $F_1(\theta)$ has everywhere regular discontinuities, and $\lim_{r=1} F_1(r, \theta) = F_1(\theta)$ for all θ.*

18. Proof of fundamental theorem. We may now prove the first part of the fundamental theorem. Since $F(r, \theta) = F_1(r, \theta) - F_1(r, 0)$, if we define $F(\theta) = F_1(\theta) - F_1(0)$, we have for all θ

$$\lim_{n \to \infty} F(r_n, \theta) = F(\theta).$$

But also the total variation of $F(r_n, \theta)$ is bounded for all n. Fix then an arbitrary point $M = (r, \theta)$, $r < 1$. We have

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(r_n^2 - r^2)}{r_n^2 + r^2 - 2 r_n r \cos(\varphi - \theta)} dF(r_n, \varphi), \quad r_n > r.$$

Here however, since r is fixed,

$$\left| \frac{1-r^2}{1+r^2-2r\cos(\varphi-\theta)} - \frac{r_n^2-r^2}{r_n^2+r^2-2r_n r\cos(\varphi-\theta)} \right|$$

is uniformly small, for all φ, with $1-r_n$. Hence we have the hypotheses of Corollary 3 of Theorem 3, Chap. I, and

$$\lim_{n=\infty} \frac{1}{2\pi}\int_0^{2\pi} \frac{(r_n^2-r^2)\,dF(r_n,\varphi)}{r_n^2+r^2-2r_n r\cos(\varphi-\theta)}$$
$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)\,dF(\varphi)}{1+r^2-2r\cos(\varphi-\theta)}.$$

Consequently $u(r,\theta)$ is given by this last integral, which is (A). Thus the point is proved.

Corollary 2. *The conditions that $u(r,\theta)$ be the difference of two not negative harmonic functions within the circle is sufficient as well as necessary for the Poisson-Stieltjes formula* (A).

In fact, if $u(r,\theta) = u'(r,\theta) - u''(r,\theta)$ where u', u'' are not negative within the circle, and harmonic, the mean value theorem for harmonic functions tells us that

$$\int_0^{2\pi} |u(r,\theta)|\,d\theta \leqq (u'_{r=0} + u''_{r=0})\,2\pi, \qquad r<1.$$

But this condition we have just proved to be sufficient for (A). The necessity of the condition in the corollary has already been proved. It is not assumed of course that $u(r,\theta)$ is bounded.

As an application of this corollary, consider the following case. Let there be an arbitrary distribution of positive and negative masses, each finite in total amount, outside or on the circumference of the unit circle, but in a finite region, where the law of attraction of point masses gives a force $-kmm'/r$, r being the distance between them. Since the potential due to the totality of positive masses is a constant plus a not negative function, and that due to the totality of negative masses is a constant plus a not positive function inside the circle, the conditions of the corollary are met with.

and *the function inside the circle may be represented by a Poisson Stieltjes integral on the boundary.**

Let us now return to the second part of the fundamental theorem. Since (ii) implies (i), the $u(r, \theta)$ is given by a formula (A), in which the $F(\theta)$ may be chosen so as to have regular discontinuities if any. If by $f(\theta)$ we denote the unique derivative of $F(\theta)$ where it exists, $f(\theta)$ will be summable and the theorem of de la Vallée Poussin tells us that in virtue of (ii) we shall have

$$\lim_{n=\infty} \int_0^\theta u(r_n, \theta)\, d\theta = \int_0^\theta f(\theta)\, d\theta.$$

However, in virtue of (4),

$$\lim_{n=\infty} \int_0^\theta u(r_n, \theta)\, d\theta = F(\theta) - F(0).$$

Hence $F(\theta) - F(0) = \int_0^\theta f(\theta)\, d\theta$; $F(\theta)$ is absolutely continuous, and (A) reduces to (B). This is what we wished to prove.

We say that $f_r(x)$ converges in the mean of the first order to $f(x)$ in an interval (a, b) as r approaches 1 if

$$\lim_{r=1} \int_a^b |f(x) - f_r(x)|\, dx = 0.$$

In terms of this concept we have the following proposition.

COROLLARY 3. NOAILLON'S THEOREM.† *Given* $f(\theta)$ *summable in the Lebesgue sense, a necessary and sufficient condition that* $u(r, \theta)$, *harmonic inside the unit circle, be given by* (B) *is that* $u(r, \theta)$, *considered as a function of* θ *in the interval* $(0, 2\pi)$ *converge in the mean of the first order to* $f(\theta)$ *as* r *tends to* 1.

* The constant may be taken as $\pm(\log R)\left(\sum |m|\right)$ where R is the diameter of the set of points on which the mass is distributed.

† Comptes Rendus, t. 182 (1926), p. 1371. Compare Theorem XIV in Kellogg's paper [*Harmonic functions and Green's integral*, Transactions of the American Mathematical Society, vol. 13 (1912), p. 132], which in a sense is a special case of those given in this chapter.

That the condition is necessary follows from the uniform absolute continuity of the integral $\int |u(r, \theta) - f(\theta)| \, d\theta$ when $u(r, \theta)$ is given by (B). By de la Vallée Poussin's theorem, then, we have

$$\lim_{r=1} \int_0^{2\pi} |u(r, \theta) - f(\theta)| \, d\theta = 0.$$

That the condition is sufficient comes from the fact that

$$\int_0^{2\pi} |u(r, \theta)| \, d\theta \leqq \int_0^{2\pi} |f(\theta)| \, d\theta + \int_0^{2\pi} |u(r, \theta) - f(\theta)| \, d\theta$$

and therefore remains bounded. Hence $u(r, \theta)$ is given by a formula (A) in which $F(\theta) = \int_0^{2\pi} f(\theta) \, d\theta$, that is, by (B).

It is desirable to write this convenient condition in a form which does not employ the boundary values $f(\theta)$, but merely the values $u(r, \theta)$, $r < 1$. We say that $u(r, \theta)$ converges in the mean of the first order, as r approaches 1, in the interval $(0, 2\pi)$, if

$$\lim_{\substack{r'=1 \\ r''=1}} \int_0^{2\pi} |u(r'', \theta) - u(r', \theta)| \, d\theta = 0$$

however r' and r'' approach 1; that is, if given ε arbitrarily we can find $r_\varepsilon < 1$ so that

$$\int_0^{2\pi} |u(r'', \theta) - u(r', \theta)| \, d\theta < \varepsilon$$

for all values of r', r'' in the open interval $(r_\varepsilon, 1)$.*

We know however that in this case (as with convergence in the mean of the second order) that there is one and only one function $f(\theta)$, except for definition on an arbitrary set of values of θ, of zero measure, such that $u(r, \theta)$ converges in the mean of the first order to $f(\theta)$.† Moreover $f(\theta)$ is summable. Hence we have immediately

* This concept is used in even more fundamental fashion by H. E. Bray in a memoir, written in conjunction with the author of the present monograph, in press, on the generalized Dirichlet problems for the sphere.

† See Chap. VII, Art. 49.

COROLLARY 4. *A necessary and sufficient condition that $u(r, \theta)$, harmonic within the unit circle, be given by a Poisson integral* (B) *is that $u(r, \theta)$ converge in the mean of the first order as r approaches* 1.

19. Special cases of the Poisson integral (B). There are several particular classes of functions given by (B), which it is worth while to specify. In the first place we have *the class of functions harmonic and bounded, $r < 1$.*

In this case, if $|u(r, \theta)| \leqq M$

$$\int_e |u(r, \theta)|\, d\theta \leqq M \cdot (\text{meas } e) \tag{4'}$$

so that the absolute continuity of the integral $\int u(r, \theta)\, d\theta$ is uniform for all $r < 1$. Hence in this case $u(r, \theta)$ is given by a formula (B) where $|f(\varphi)|$ is bounded (Of course $f(\varphi)$ may be assigned arbitrarily on a set of values of φ of measure 0, without affecting the value of the integral (B)). Conversely, *if $f(\varphi)$ is bounded and integrable in the Lebesgue sense,* the $u(r, \theta)$ will be bounded, $r < 1$.

Secondly, we have the class of functions, which includes the former class, where $u(r, \theta)$, *harmonic within the circle, satisfies a condition of the form*

$$|u(r, \theta)| \leqq \psi(\theta), \qquad r < 1, \tag{5}$$

where $\psi(\theta)$ is summable in the Lebesgue sense.

In this case,

$$\int_{\theta'}^{\theta''} |u(r, \theta)|\, d\theta \leqq \int_{\theta'}^{\theta''} \psi(\theta)\, d\theta \tag{6}$$

and therefore it follows immediately that the absolute continuity of the integral $\int |u(r, \theta)|\, d\theta$ is uniform for all $r < 1$. Hence (5) is a sufficient condition for (B).

Consider as a third case the class of harmonic functions such that the numerical value of the derivative with respect to r is summable over the region bounded by the circle.

The quantity $(1/r)\dfrac{\partial u}{\partial r}$ will also be summable over the same region. Consider then regions $\sigma' = \sigma(r_0, r_1; \theta', \theta'')$ and

$\sigma = \sigma(r_0;\ \theta',\ \theta'')$ where σ' is the region $r_0 \leqq r \leqq r_1$; $\theta' \leqq \theta \leqq \theta''$, and σ is the region $r_0 \leqq r < 1$; $\theta' \leqq \theta < \theta''$. We have

$$\int_{\sigma'} \frac{1}{r} \frac{\partial u}{\partial r} \cdot d\sigma = \int_{\theta'}^{\theta''} d\theta \int_{r_0}^{r_1} \frac{\partial u}{\partial r} \cdot dr$$

$$= \int_{\theta'}^{\theta''} [u(r_1, \theta) - u(r_0, \theta)]\, d\theta$$

and therefore

$$\left| \int_{\theta'}^{\theta''} u(r_1, \theta)\, d\theta \right| \leq \int_{\theta'}^{\theta''} | u(r_0, \theta) |\, d\theta + \int_{\sigma'} \left| \frac{1}{r} \frac{\partial u}{\partial r} \right| d\sigma$$

$$\leqq \int_{\theta'}^{\theta''} | u(r_0, \theta) |\, d\theta + \int_{\sigma} \left| \frac{1}{r} \frac{\partial u}{\partial r} \right| d\sigma.$$

A continuous function is positive or zero on at most a denumerable infinity of closed intervals. If accordingly we take the denumerable set of closed intervals in (θ', θ'') where $u(r_1, \theta) \geqq 0$, constituting a set e', and the corresponding denumerable set of angular portions of σ constituting a set E', the above inequality yields the relation

$$\int_{e'} | u(r_1, \theta) |\, d\theta \leqq \int_{e'} | u(r_0, \theta) |\, d\theta + \int_{E'} \left| \frac{1}{r} \frac{\partial u}{\partial r} \right| d\sigma.$$

In a similar fashion, for the set of open intervals e'' in (θ', θ'') where $u(r_1, \theta) < 0$, and the corresponding set E'', we have

$$\int_{e''} | u(r_1, \theta) |\, d\theta \leqq \int_{e''} | u(r_0, \theta) |\, d\theta + \int_{E''} \left| \frac{1}{r} \frac{\partial u}{\partial r} \right| d\sigma.$$

But e'' and e' have no points in common, likewise E'' and E'. Therefore we have the inequality

$$(7) \qquad \int_{\theta'}^{\theta''} | u(r_1, \theta) |\, d\theta \leqq \int_{\theta'}^{\theta''} | u(r_0, \theta) |\, d\theta + \int_{\sigma} \left| \frac{1}{r} \frac{\partial u}{\partial r} \right| d\sigma.$$

But the integrals of the right hand member of (7) are absolutely continuous functions of θ'' which do not involve r_1. It follows that the absolute continuity of the left hand member of (7) is uniform for all $r_1 < 1$. Hence this third case also gives us functions represented by the formula (B).

EXERCISE. Show that the third case is included in the second.

An interesting type of functions covered by the third case is that where

$$\int\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]d\sigma$$

exists when the region of integration is the circle. In fact

$$\int\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right]d\sigma=\int\left[\left(\frac{\partial u}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2\right]d\sigma$$
$$\geqq\int\left(\frac{\partial u}{\partial r}\right)^2\partial\sigma,$$

and if the last integral exists, then also $\int\left|\frac{\partial u}{\partial r}\right|d\sigma$ exists.

20. The Dirichlet problem and its extension. The formula (B) yields the solution of the Dirichlet problem.

THEOREM 2. *Given the function $f(\theta)$, summable in the Lebesgue sense there is one and only one function $u(r,\theta)$ harmonic within the circle, of the class defined by* (ii), *such that* $\lim_{r=1} u(r,\theta)=f(\theta)$ *almost everywhere. In particular, if $f(\theta)$ is bounded $u(r,\theta)$ is bounded, and conversely.*

The formula (A) yields the solution of a generalized Dirichlet problem.

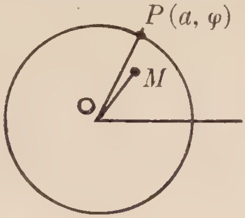
THEOREM 3. *Given the function $F(\theta)$, of limited variation, there is one and only one function $u(r,\theta)$ harmonic within the circle, of the class defined by* (i), *such that* $\lim_{r=1}\int_{\theta_1}^{\theta_2}u(r,\theta)\,d\theta$ $=F(\theta_2)-F(\theta_1)$, *for θ_1, θ_2 in a set of values of θ dense on the circumference. In fact this boundary condition is satisfied except for values of θ_1 or θ_2 which correspond to discontinuities of $F(\theta)$, at most denumerable in number, when these are not regular.*

EXERCISE. State the solution of the Dirichlet problem in terms of Noaillon's theorem.

CHAPTER IV

POTENTIALS OF A SINGLE LAYER AND THE NEUMANN PROBLEM

21. The Stieltjes integral for potentials of a single layer.* Consider the integral



$$(C)\quad v(M) = \frac{a}{\pi}\int_0^{2\pi} \log\frac{1}{MP}\, dF(\varphi_P) + A,$$

or, what is the same thing,

$$(1)\quad v(r, \theta) = -\frac{a}{2\pi}\int_0^{2\pi} \log\left(a^2 + r^2 - 2ar\cos(\varphi - \theta)\right) \times dF(\varphi) + A,$$

where A is a constant. This represents the potential due to the most general distribution of simple masses on the circumference of a circle of radius a if $F(\varphi)$ is the most general function of limited variation. We notice however, that for the particular case $F(\varphi) = c\varphi$ the potential (C) reduces to a constant.

In fact, for $\varrho < 1$,

$$\log\sqrt{1 + \varrho^2 - 2\varrho\cos\varphi} = -\varrho\cos\varphi - \varrho^2\frac{\cos 2\varphi}{2} - \varrho^3\frac{\cos 3\varphi}{3} - \cdots$$

and so for $r < a$

$$\log\sqrt{a^2 + r^2 - 2ar\cos(\varphi - \theta)} = \log a - \frac{r}{a}\cos(\varphi - \theta) - \frac{r^2}{a^2}\cos\frac{2(\varphi - \theta)}{a} - \cdots.$$

Hence

$$-\frac{a}{2\pi}\int_0^{2\pi} \log\left(a^2 + r^2 - 2ar\cos(\varphi - \theta)\right) dc\varphi = -ac\log a^2, \qquad r < a,$$

which we wished to prove.

* My predecessor in this point of view is Plemelj, *Potentialtheoretische Untersuchungen*, Leipzig (1911).

Exercise. Deduce the above development in terms of ϱ and φ, as the real part of a function of a complex variable.

The constant which we have just found may be included in the constant A, and therefore there is no loss of generality in the form (C) if we suppose that $F(\varphi)$ is periodic, of period 2π. Moreover the value of the Stieltjes integral for $r < 1$ does not depend on the particular values which $F(\varphi)$ has at discontinuities, provided it remains of limited variation. We may accordingly regard the discontinuities as *regular*, that is, of the type of the Fourier representation of the function. In this way the function $F(\varphi)$ will be uniquely determined by the values of $v(r, \theta)$ at interior points of the circle, as we shall see.

22. Necessary and sufficient condition. Theorem 1. *A necessary and sufficient condition in order that $v(r, \theta)$, harmonic inside the circle of radius a, be given by an integral of type (C) is that there should be a constant N, such that*

$$\int_0^{2\pi} \left| \frac{\partial v(r, \theta)}{\partial r} \right| d\theta < N, \qquad r < a. \tag{2}$$

Since the $F(\varphi)$ is periodic, (1) is equivalent, by an integration by parts, to the following:

$$v(r, \theta) = \frac{a}{2\pi} \int_0^{2\pi} \frac{2ar \sin(\varphi - \theta)}{a^2 + r^2 - 2ar\cos(\varphi - \theta)} F(\varphi)\, d\varphi + A,$$

which represents the function conjugate to the function

$$u(r, \theta) = -\frac{a}{2\pi} \int_0^{2\pi} \frac{F(\varphi)(a^2 - r^2)}{a^2 + r^2 - 2ar\cos(\varphi - \theta)} F(\varphi)\, d\varphi + B,$$

where B is a constant.* Perform now on this integral a differentiation and follow it by an integration by parts,

* Osgood, *Funktionentheorie*, Leipzig (1912), vol. 1, p. 634. The formula is obtained by finding the imaginary part of $\frac{1+z}{1-z}$ (See Art. 11).

keeping account of the periodicity of $F(\varphi)$. We obtain the following equation

$$(3)\quad r\frac{\partial v}{\partial r} = -\frac{\partial u}{\partial \theta} = \frac{a}{2\pi}\int_0^{2\pi}\frac{(a^2-r^2)\,dF(\varphi)}{a^2+r^2-2\,ar\cos(\varphi-\theta)}.$$

Conversely, from (3) we have (1), since $v(r,\theta)$ is a single valued harmonic function within the circle.

But we have already seen that a necessary and sufficient condition for a representation of type (3) is that the harmonic function to be so represented satisfy a certain integral condition—the condition (i), Chap. III,—which in this case becomes

$$\int_0^{2\pi}\left|r\frac{\partial v}{\partial r}\right|d\theta < N', \qquad N' \text{ independent of } r.$$

And this is equivalent to the condition of the theorem to be proved.

23. Further properties. If we proceed to apply the results already obtained for (3) to the problem in hand, we obtain the following facts:

$$(4)\qquad \lim_{r=a}\frac{\partial v(r,\theta)}{\partial r} = F'(\theta),$$

for every θ for which $F'(\theta)$ exists and is unique, and that is almost everywhere on the circumference of the circle. Here the manner of approach of (r, θ_r) to (a, θ) is in the wide sense of Theorem 7, Chap. II, which includes the particular case of $\theta_r = \theta$, where the point approaches the boundary along a radius.

$$(5)\qquad \lim_{r=a}\int_0^{\theta}\frac{\partial v}{\partial r}\,d\theta = F(\theta)-F(0),$$

for every θ.

(j) A necessary and sufficient condition for (1) is that $v(r,\theta)$ be the difference of two functions each harmonic at interior points of the circle, except at the center, where both functions have the same logarithmic singularity, and each is non-decreasing as a function of r.

(jj) A necessary and sufficient condition that $F(\varphi)$ be absolutely continuous, that is, the integral of its derivative $F'(\varphi) = f(\varphi)$, is that the absolute continuity of the integral $\int \frac{\partial v}{\partial r} d\theta$ be uniform, $r < a$.

THEOREM 2. *Under the condition* (jj), *the* $v(r, \theta)$ *is given by the formula*

$$(D) \quad v(r, \theta) = \frac{a}{2\pi} \int_0^{2\pi} \log \frac{1}{a^2 + r^2 - 2ar\cos(\varphi - \theta)} f(\varphi)\, d\varphi + A,$$

where $\int_0^{2\pi} f(\varphi)\, d\varphi = 0$, *by means of the proper choice of* A. *In fact,* A *is the value of* $v(r, \theta)$ *when* $r = 0$.

24. The Neumann problems. Equation (D) gives the solution of the Neumann problem for the class of functions (jj), the values $f(\varphi)$ which the normal derivative is to take on at the boundary almost everywhere being arbitrary provided that they are summable in the Lebesgue sense and such that their integral around the boundary is zero (if the integral of the boundary values of $\frac{\partial v}{\partial r}$ around the circumference were not zero, the function v could not be finite inside). A particular class of the functions (jj) are those for which $\left|\frac{\partial v}{\partial r}\right|$ remains bounded, $r < a$. For these functions $f(\varphi)$ is also bounded, except of course on an arbitrary set of values of φ of zero measure.

Equation (C) solves a generalized Neumann problem. This rather than the other is essentially the physical problem of the Neumann type.

THEOREM 3. *Among the functions whose total absolute flux* $\int_0^{2\pi} \left|\frac{\partial v}{\partial r}\right| d\theta$ *is bounded,* $r < a$, *there is one and only one, except for an arbitrary additive constant, which satisfies the equation* (5),— *given* $F(\theta)$ *of limited variation, with regular discontinuities, and periodic with period* 2π.

25. General points of view. It is now an interesting question to ask if the class of functions of the theorem of

Art. 11, which gives the solution of the generalized Neumann problem is more or less general than the class of functions of condition (i) which gives the solution of the generalized Dirichlet problem. The question is answered at once. If $\int_0^{2\pi}\left|\frac{\partial v}{\partial r}\right| d\theta < N$, it follows that $\left|\frac{\partial v}{\partial r}\right|$ is summable on the region defined by the circle $r = a$, and therefore (see Art. 19) that $v(r, \theta)$ is given by a formula

$$(6) \quad v(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi}\frac{a^2 - r^2}{a^2 + r^2 - 2\,a\,r\cos(\varphi - \theta)}\cdot g(\varphi)\,d\varphi.$$

where $g(\varphi)$ is some function summable (L). But (6) is merely a special case of the Poisson-Stieltjes integral of Chapter II.

We are now faced with another question. Are the harmonic functions of what we might call *physical character* the only possible ones? Are functions which can be written inside the circle as the difference of two not negative harmonic functions the only ones possible which are harmonic within the circle? Again the answer is in the negative. There are for example those which arise from the class considered by differentiation one or more times with respect to θ. Thus there are the functions

$$(7) \qquad u(r,\theta) = a_0/2 + \sum_1^\infty r^n(a_n \cos n\theta + b_n \sin n\theta)$$

where a_n, b_n become infinite like a positive power of n. But there are also functions which are not derivatives with respect to θ, of any finite order, of functions of *physical character*. Consider for example a function given by (7) when the a_n, b_n become infinite like the quantity $\alpha^{\sqrt{n}}$, $\alpha > 1$.*

26. Digression. Physical interpretation of a general distribution of mass. It has already been pointed out in

* The extension of this treatment to spaces of dimension > 2 constitutes one of the most interesting applications of Volterra's theory of functions of curves and integral equations. In fact the limits of the absolutely continuous functions which must be considered are functions of curves which are additive, but which are not additive functions of point sets.

Art. 13 that the function $F(\varphi)$ corresponds to an arbitrary distribution of positive and negative mass, $\Phi(e)$, on the circumference of a circle, finite in total absolute amount. In Chapters II and III we were dealing with distributions of a double layer and in the present chapter we deal with distributions of a single layer. But we have not considered any method of constructing or laying down such a distribution $\Phi(e)$, and this is a process which is not easy to imagine. In this section we shall consider simple approximations to such arbitrary distributions, the approximations being "as close" as desired.

When we speak of an arbitrarily close approximation in a physical sense, we mean not merely that the given one is a limiting one for the approximation, but also that the physical measurements — attraction, potential, etc., from which one infers a distribution of matter, — which are given in terms of the approximation differ by as little as we please from those which belong to the given distribution.

Let P be a generic point of the circumference and M a point not on the circumference; and let $h(M, P)$ be a continuous function of M and P. The quantities which are measured depend on the whole distribution and will therefore be given by Stieltjes integrals of the form

$$\xi(M) = \int_0^{2\pi} h(M, P)\, d\, F(\varphi_P). \tag{8}$$

If then $F_n(\varphi)$ is the function of limited variation which corresponds to an approximate distribution, the corresponding measurement function will be

$$\xi_n(M) = \int_0^{2\pi} h(M, P)\, d\, F_n(\varphi_P). \tag{8'}$$

Consequently sufficient conditions for approximate measurement,

$$\lim \xi_n(M) = \xi(M), \tag{8''}$$

are given by the hypotheses of the Helly-Bray theorem.

THEOREM OF APPROXIMATE MEASUREMENT. *If* $\lim_{n=\infty} F_n(\varphi) = F(\varphi)$ *on a set of values of* φ *dense on the circumference, and* $F_n(\varphi)$ *is of uniformly limited variation, then equation* (8″) *is valid.*

Incidentally, all functions of limited variation which correspond to the same additive function of point sets $\Phi(e)$ (that is, to the same distribution of matter) yield the same value of the integral $\xi(M)$, for any two such functions are extensions of each other as defined on a common set E where both are continuous, except for an arbitrary additive constant. Thus if there is a point charge at φ_0 its value is always $F(\varphi_0+0)-F(\varphi_0-0)$.

Any function of limited variation $F(\varphi)$ may be written as the sum of three terms of different kinds

$$(9) \qquad F(\varphi)-F(0) = \alpha(\varphi)+\int_0^{\varphi} f(\varphi)\,d\varphi+\lambda(\varphi).$$

In this representation $\alpha(\varphi)$ represents the function of discontinuities; i. e., if the discontinuities are at points $\varphi_1, \varphi_2, \cdots$, taken in some denumerable order,

$$(10) \qquad \begin{aligned} \alpha(\varphi) &= \sum_{0<\varphi_n\leqq\varphi}\{F(\varphi_n)-F(\varphi_n-0)\} \\ &\quad + \sum_{0\leqq\varphi_n<\varphi}\{F(\varphi_n+0)-F(\varphi_n)\}, \text{ for } 0<\varphi\leq 2\pi, \\ &= 0, \text{ for } \varphi=0, \end{aligned}$$

where the summations are extended over the discontinuities φ_n which are indicated. We then write $f(x)$ as the derivative, where it exists, of the continuous function $F(\varphi)-F(0)-\alpha(\varphi)$, and define it arbitrarily otherwise; in fact, $f(\varphi) = F'(\varphi)$ almost everywhere. What is left, namely

$$(11) \qquad \lambda(\varphi) = F(\varphi)-F(0)-\alpha(\varphi)-\int_0^{\varphi} f(\varphi)\,d\varphi,$$

is a continuous function of limited variation with a zero derivative almost everywhere.

Vitali has shown that $\lambda(\varphi)$ may be written in the form

$$(12) \qquad \lambda(\varphi) = \sum_1^\infty k_i \psi_i(\varphi),$$

where $\sum |k_i|$ is convergent, and each of the functions $\Psi(\varphi) = \psi_i(\varphi)$ (called an *elementary discard*) is continuous, non-decreasing (with $\Psi(0) = 0$, $\Psi(2\pi) = 1$) and constant on each interval of a denumerable sequence of intervals $s_1, s_2, \cdots$ (called a *plurisegment*) of which the complementary set E_Ψ is perfect and of zero measure.* According to Vitali a denumerable set of values y_i can be found, dense in (0, 1) such that $\Psi(\varphi) = y_i$ on the interval s_i, and $y_k > y_m$ if s_k is to the right of s_m. We might call the distribution which is given by $\Psi(\varphi)$ an *elementary discard distribution.*

The distribution given by $F(\varphi) - F(0)$ is the sum of those given by its three parts. In the case of $\alpha(\varphi)$ the corresponding distribution may be interpreted as the sum of those represented by its separate terms; in fact, the function corresponding to the sum of the first p terms (following a denumerable order) has $\alpha(\varphi)$ as its limit for all values of φ, and its total variation is uniformly bounded, for all p, not exceeding the total variation of $\alpha(\varphi)$. Hence the requirements for the theorem of approximate measurement are fulfilled. Each of the single terms represents a point charge.

Similarly the distribution represented by $\lambda(\varphi)$ may be regarded as formed by successively adding the distributions corresponding to the separate terms $k_i \psi_i(\varphi)$. In fact the function

$$(13) \qquad \lambda_p(\varphi) = \sum_1^p {}_i k_i \psi_i(\varphi)$$

has $\lambda(\varphi)$ for its limit for all values of φ; moreover the total variation of $\lambda_p(\varphi)$ is $\leqq \sum_1^p |k_i| \leqq \sum_1^\infty |k_i|$, and is therefore uniformly bounded. Hence again the conditions of the measurement theorem are satisfied. We are thus left with three types of distribution to interpret, viz., the elementary discard

* Rendiconti del Circolo Matematico di Palermo, vol. 46 (1922), p. 388. The illustration in Art. 2 above is the graph of an elementary discard.

distribution, the point charge and the absolutely continuous function, the last being a charge which is the integral of its density over any interval, and therefore the kind of charge usually considered in integral formulae.

As a first interpretation let us consider the elementary discard as the limiting form of a distribution spread uniformly (that is, with constant density) over each of a finite number of intervals. To be precise, let $\Psi_n(\varphi)$ be a polygonal approximation to $\Psi(\varphi)$, formed by joining successively with straight lines a finite number of points on the graph of $\Psi(\varphi)$, including for convenience those for 0 and 2π. The approximation $\Psi_{n+1}(\varphi)$ is formed by retaining the vertices of $\Psi_n(\varphi)$ and inserting new ones between them, in such a way that the vertices of all the polygons taken together form a set E of values of φ dense in $(0, 2\pi)$. In this way the approximate distribution consists of uniform patches on the circumference of the circle, and the successive approximations subdivide these into smaller patches, only those which contain points of the perfect set E_Ψ possessing any charge. The perfect set E_Ψ is in fact obtained by cutting out intervals.

The total variation of $\Psi_n(\varphi)$ obviously is precisely unity, and is therefore bounded uniformly, and $\lim\limits_{n \to \infty} \Psi_n(\varphi) = \Psi(\varphi)$ for the points of E. The equation (8″) is therefore valid for the approximation.

Instead of a polygonal approximation we may use a step-function as $\Psi_n(\varphi)$, based on the same points of division, the graph now being composed of a finite number of horizontal lines, and discontinuous. The corresponding approximate distribution consists of a finite number of point masses, and for further approximations these are successively subdivided and shifted (the portions of any mass for a given approximation always remaining, however divided, within two contiguous intervals of that approximation) until they are located on the set E_Ψ of zero measure.

The absolutely continuous function $\int_0^\varphi f(\varphi)\,d\varphi$ may similarly be interpreted either with the help of a polygonal approxima-

tion as the limit of a distribution uniform in patches, or with the help of a step function as the limit of a distribution of point charges. The point charge may also be regarded as the limit of a uniform patch. In fact there is nothing to prevent us from applying these methods directly to the function $F(\varphi)$ itself; we know that as long as we form a limiting distribution which agrees with $\Phi(e)$ on all intervals with end points in E, it must be identical with $\Phi(e)$ itself. This is of course not the same as saying that $\lim_{n \to \infty} \Phi_n(e) = \Phi(e)$ for all measurable sets e. It may be remarked however that in dealing directly with $F(\varphi)$ we mask the nature of the particular kinds of distributions (like elementary discards) out of which $\Phi(e)$ is built.

Other kinds of approximations are also available. We might for instance have used the Fejér trigonometric sums, which converge to $F(\varphi)$ except at its discontinuities and are of uniformly limited variation, as was seen in Chap. I. They yield approximate distributions in which the density varies continuously, but lack the concreteness of the other interpretations suggested.

We are not able, merely by describing possible distributions of mass, to infer their physical existence. On the other hand, apparent resolution of matter into components of one kind does not preclude the existence of other kinds. To take a concrete example in the field of energy, it may be remarked that although band spectra seem to be resolvable indefinitely into sequences of fine lines, it may be that physicists, by means of that resolution, are dealing with an approximation by points to distributions of energy which would ultimately be more conveniently interpreted in terms of elementary discards. Whether we arrive at point distributions as a theoretical explanation or elementary discards, apparently hinges on the question as to whether it is the intervals of wave length in which distributions occur, which are most important, or the intervals from which they are excluded.

The results obtained in this section have been stated in physical terms. They may also, if we take for the function

$h(M, P)$ the two special forms which we have used in the last three chapters, be regarded as the statement of closure properties for certain classes of harmonic functions.

27. Cauchy's integral formula. Let $u(r, \theta)$ be a harmonic function of class (ii) inside the circle of radius a, so that it is given by a Poisson integral of type (B)

$$u(M) = u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(a^2-r^2)\, f(\varphi)}{a^2+r^2-2\,ar\cos(\varphi-\theta)}\, d\varphi$$

in which $f(\varphi)$ is summable in the Lebesgue sense. This function is the real part (see Art. 11) of the function

$$\begin{aligned} w(z) &= u + iv \\ &= \frac{1}{2\pi}\int_0^{2\pi} \frac{t+z}{t-z}\, f(\varphi)\, d\varphi + \text{const.}, \\ &= -\frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\, d\varphi + \frac{1}{\pi}\int_0^{2\pi} \frac{t}{t-z}\, f(\varphi)\, d\varphi + \text{const.}, \end{aligned}$$

where we write $t = ae^{i\varphi}$, $z = re^{i\theta}$. If we put $z = 0$ in this equation we find the value of the constant to be $iv(0)$, so that the equation becomes the following, when we write $dt = it\, d\varphi$ and $f(t)$ for the corresponding $f(\varphi)$,

$$w(z) = -u(0) + iv(0) + \frac{1}{\pi i}\int_c \frac{f(t)\, dt}{t-z}, \tag{14}$$

with

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} f(\varphi)\, d\varphi. \tag{14'}$$

We are able therefore to state the following lemma.

LEMMA. That $u(r, \theta)$ be a harmonic function of class (ii) inside the circle of radius a is a necessary and sufficient condition that $w(z)$ satisfy an equation of form (14), where $f(t)$ is a real-valued function summable on the circumference and $u(0)$ is given by (14′). The real part of $w(z)$ takes on the boundary values $f(t)$ almost everywhere, as z approaches t in the wide sense, and $w(z)$ is uniquely determined by those values except for the arbitrary imaginary constant $iv(0)$.

Suppose now that both u and v are of class (ii), for which a necessary and sufficient condition is that $|u+iv|$ be of class (ii). In this case we shall say that $w(z)$ is of class (ii). Then (14) holds; but also, similarly,

$$\begin{aligned} -iw(z) &= v(r,\theta)-iu(r,\theta) \\ &= -v(0)-iu(0)+\frac{1}{\pi i}\int_c \frac{g(t)\,dt}{t-z} \end{aligned}$$

or

$$w(z) = u(0)-iv(0)+\frac{1}{\pi i}\int_c \frac{ig(t)\,dt}{t-z} \tag{15}$$

where $g(t)$ is a summable real-valued function. From (14) and (15) we deduce the equation

$$w(z) = \frac{1}{2\pi i}\int_c \frac{f(t)+ig(t)}{t-z}\,dt. \tag{16}$$

In other words, $w(z)$ may be given by a Cauchy integral formula (16) where $\lim_{z \to t} w(z) = f(t)+ig(t)$ almost everywhere, in the wide sense.

But the real-valued functions f and g, as developed in (16), are not independent. In order to state the relations between them in convenient form let us introduce the Fourier coefficients a'_m, b'_m of $f(\varphi)$ and a''_m, b''_m of $g(\varphi)$, quantities which we know to exist since the two functions are summable. In fact, from the convergence of the Fejér summation method, we know that these summable functions are determined almost everywhere by their Fourier coefficients. We may state then the following theorem.

Theorem 3. *A necessary and sufficient condition that $w(z)$ may be written in terms of a Cauchy integral formula* (16), *with summable real-valued functions $f(t)$, $g(t)$, is that $w(z)$ be analytic inside the circle and of class* (ii). *In order that $w(z)$, so given, take on the boundary values $f(t)+ig(t)$ almost everywhere in the wide sense, it is necessary and sufficient that*

$$a'_m = b''_m, \qquad b'_m = -a''_m, \qquad m = 1, 2, \cdots, \tag{17}$$

where a'_m, b'_m are Fourier coefficients of $f(\varphi)$ and a''_m, b''_m are Fourier coefficients of $g(\varphi)$.

The functions $f(\varphi)$, $g(\varphi)$ are therefore determined in terms of each other except for arbitrary additive constants.

It is only the second part of the theorem which requires further proof. We write $w(z) = w_1(z) + w_2(z)$ where

$$\begin{aligned} w_1(z) &= u_1(r, \theta) + i v_1(r, \theta) \\ &= \frac{1}{\pi i}\int_c \frac{\frac{1}{2}f(t)}{t-z}\,dt - \frac{1}{2\pi}\int_0^{2\pi} \frac{1}{2}\{f(\varphi) - ig(\varphi)\}\,d\varphi \end{aligned} \tag{18}$$

$$\begin{aligned} w_2(z) &= u_2(r, \theta) + i v_2(r, \theta) \\ &= \frac{1}{\pi i}\int_c \frac{\frac{i}{2}g(t)}{t-z}\,dt + \frac{1}{2\pi}\int_c^{2\pi} \frac{1}{2}\{f(\varphi) - ig(\varphi)\}\,d\varphi, \end{aligned} \tag{19}$$

so that (18) and (19) become special cases of (14) and (15) respectively, and, almost everywhere,

$$\lim_{r=1} u_1(r, \theta) = \frac{f(\theta)}{2}, \qquad \lim_{r=1} v_2(r, \theta) = \frac{i}{2} g(\theta),$$

Hence, by the lemma,

$$w(z) = 2w_1(r, \theta) + Ai = 2w_2(r, \theta) + B$$

where A and B are real constants. But putting $z = 0$ in (18) and (19) we have

$$w(0) = \begin{cases} 2w_1(0) + Ai = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\{f(\varphi) + ig(\varphi)\}\,d\varphi + Ai \\[2ex] 2w_2(0) + B \quad \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\{f(\varphi) + ig(\varphi)\}\,d\varphi + B, \end{cases}$$

so that $A = B = 0$, and $w_1(z) \equiv w_2(z)$.

This last condition may be written in the form

$$\frac{1}{2\pi i}\int_c \frac{f(t) - ig(t)}{t-z}\,dt + \frac{1}{2\pi}\int_0^{2\pi}\{-f(\varphi) + ig(\varphi)\}\,d\varphi = 0, \tag{20}$$

and by substition in (18) and (19) is seen to be sufficient, as well as necessary, that $w(z)$ take on the required boundary values. The left hand member of (20) however represents an analytic function of z, and for this function to be identically zero it is necessary and sufficient that it should vanish with all its derivatives, at the point 0.

The equation is already satisfied when $z = 0$. Hence the required conditions are merely the following,

$$\int_c \frac{f(t) - i g(t)}{t^k}\, dt = 0, \quad k = 2, 3, \cdots,$$

or, finally, when we have written $t = e^{\varphi i}$, $dt = i t d\varphi$,

$$\int_0^{2\pi} \{f(\varphi) - i g(\varphi)\}\{\cos m\varphi - i \sin m\varphi\}\, d\varphi = 0, \quad m = 1, 2, \cdots.$$

which are precisely the equations (17), Hence the theorem is proved.

Corollary 1. Given the real function $f(\varphi)$ summable with its square, whose Fourier coefficients a'_m, b'_m, are such that the series

$$\sum_1^\infty (a'^2_m + b'^2_m)(\log m)^2$$

converges, there is a real function $g(\varphi)$, also summable with its square, uniquely determined by (17) except for an arbitrary constant. These two functions are given by their Fourier series, and the function $w(z)$, given by (16) takes on almost everywhere the boundary values $f(t) + ig(t)$ in the wide sense.

This corollary is verified at once by using Theorem 5, Chap I. With the aid of the Fischer-Riesz theorem we have a slightly more general result.*

Corollary 2. Given $f(\varphi)$, real and summable with its square, $g(\varphi)$ and $w(z)$ are determined so that the conclusions of Corollary 1 follow, except that $f(\varphi)$ and $g(\varphi)$, instead of

*A proof of the Fischer-Riesz theorem is given in Hobson, *Theory of functions of a real variable*, vol. II, Cambridge (1926), p. 576.

being given by their Fourier series, are given by the Fejer summation process.

EXERCISE 1. With the help of (18) and (19) discuss the integral (16) when $f(t) = ig(t) = 1/t$.*

EXERCISE 2. We shall say that $w(z)$ converges in the mean of the first order as z approches the boundary if $\lim_{r=1} \int_c |w(z') - w(z'')|\,|dz| = 0$, where z' denotes the point (r', θ), z'' the point (r'', θ) and $r \leqq r' < r'' < 1$. Show that this condition is necessary and sufficient that $w(z)$, analytic inside the circle, may be given by (16) (see Art. 18).

EXERCISE 3. Discuss the equation

$$\frac{w(z) - w(0)}{z} = \frac{1}{2\pi} \int_c \frac{d\Psi(t)}{t - z},$$

where $\Psi(t) = F(\varphi) + iG(\varphi)$, these functions of φ being real and of limited variation.

EXERCISE 4. Show that if $u(r, \theta)$ is given, harmonic within the circle, such that $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$ is summable over the circle, the analytic function $u + iv = w$ may be given by a formula (16).

EXERCISE 5. If $u(r, \theta)$ is given by a formula (C), Art. 21, and is of uniformly limited variation in θ for all $r < 1$, show that $w = u + iv$ may be given by a formula (16). The circle is transformed by the corresponding conformal transformation into a region with rectifiable boundary.†

EXERCISE 6. If $w(z)$ is given by a formula (16) in which $f(t)$ and $g(t)$ are real and summable on the circumference, but do not satisfy (17), it may also be given by such a formula where (17) will be satisfied.

* See Osgood, *Funktionentheorie*, Leipzig (1912), p. 297.

† It is clear that the boundary functions $f(\varphi)$ and $g(\varphi)$ are of limited variation. It is desired to show that they are continuous. Suppose that $f(\varphi)$ has a positive jump at $\varphi = 0$, such that $f(0+) - f(0-) = A$. Then, since the discontinuity is regular, we have for the positive and negative variation functions, $\varphi(0+) - \varphi(0-) = A$, $\psi(0+) - \psi(0-) = 0$.

Now, $v(r, \theta)$ being conjugate to $u(r, \theta)$, we have

$$v(M) = \frac{1}{\pi}\int_{-\pi}^{\pi} \log\frac{1}{MP}\, d\varphi(P) + \frac{1}{\pi}\int_{-\pi}^{\pi} \log\frac{1}{MP}\, d\psi(P) + \text{const.}$$

$$= \frac{1}{\pi}\int_{-\varepsilon}^{\varepsilon} \log\frac{1}{MP}\, d\varphi(P) + \frac{1}{\pi}\int_{-\varepsilon}^{\varepsilon} \log\frac{1}{MP}\, d\psi(P) + V(M),$$

where $M = (r, 0)$, $P = (1, 0)$ and $V(M)$ remains bounded as M approaches P along the radius. But

$$v(r, 0) \geqq \frac{A}{\pi}\log\frac{1}{1-r} - \frac{1}{\pi}\left(\log\frac{1}{1-r}\right) |\psi(\varepsilon) - \psi(-\varepsilon)| + v(r, 0),$$

the first term being the contribution to the first integral at $\varphi = 0$. But this whole expression becomes positively infinite as r tends to 1, which is contrary to the hypothesis; for $v(r, 0)$ tends to $g(0)$ which is finite. Hence $f(\varphi)$ is continuous at $\varphi = 0$, and similarly for any other φ.

The condition of Exercise 5, that $u(r, \theta)$, $v(r, \theta)$ be of limited variation as functions of θ uniformly for all r, or that $\int_0^{2\pi} |\partial u/\partial r|\, d\theta$, $\int_0^{2\pi} |\partial v/\partial r|\, d\theta$ be bounded, is necessary and sufficient that the boundary of the transformed region be a rectifiable (not necessarily simple) continuous closed curve.

CHAPTER V

GENERAL SIMPLY CONNECTED REGIONS AND THE ORDER OF THEIR BOUNDARY POINTS BOUNDARY VALUE PROBLEMS

28. Conformal transformations and general regions. Since a conformal transformation of the plane carries a harmonic function into another harmonic function, the results so far obtained will apply to regions which can be obtained from a circle by conformal transformation, in so far as those results can be stated in a form which is invariant of such transformations.

Let T be an arbitrary simply connected open region, for simplicity, in the finite plane, let C be its frontier or boundary, and let g, h be the Green's function and its conjugate respectively for T.

Then the function

$$w = f(z) = e^{-g-hi} \tag{1}$$

transforms the region T in a one-one manner and conformally on the interior of the unit circle of the w plane.

In particular, if T is bounded by a finite number of pieces of analytic curves, the transformation is also one-one for points on the boundary, and, except at the vertices, conformal.*

29. Invariant forms of conditions (i), (ii), **etc.** In the circle of radius 1, the Green's function for which the pole is at the center 0 is given by $\log 1/r$, simply, and its conjugate function is $-\theta$. Hence if we denote this Green's function for the circle by $g_0(O|M)$, and its conjugate by $h_0(O|M)$, the conditions (i), (ii) may be restated as the following ones, the integration being extended along curves $g_0(O|M) = \text{const.} = g$:

* Osgood, *Funktionentheorie*, Leipzig (1912), vol. 1, p. 682. The existence of the Green's function is an application of the theory of increasing sequences of functions to an open set. (Ibid, p. 701.)

(i) $\int_0^{2\pi} |u(M)|\, dh_0(O|M) < N, \quad g > 0,$

(ii) The absolute continuity of $\int_0^h |u(M)|\, dh_0(O|M)$ is uniform for $g > 0$.

Exercise. Show by means of a linear transformation that O may be replaced by an arbitrary interior point A' of the circle.

The conditions (i), (ii) are now obviously in a form independent of a conformal transformation. For we get equivalent conditions for the general region T merely by replacing $g(O|M)$ by $g(A|M)$ and $h(O|M)$ by $h(A|M)$, which are the Green's function and its conjugate, respectively, for T, the point A being an arbitrary fixed point of T. We have, in T, the conditions

(i) $\int_0^{2\pi} |u(M)|\, dh(A|M) < N, \quad g > 0,$

(ii) The absolute continuity of $\int_0^h |u(M)|\, dh(A|M)$ is uniform for $g > 0$,

as conditions to describe the classes of functions which arise by means of conformal transformations from the corresponding classes in the circle. This is merely an application of equation (1).

We verify at once then, also by (1), that *the class* (i) *in* T *consists of the functions which may be written as the difference of two not negative functions, harmonic in* T, and that *the class* (ii) *contains in particular the class of bounded functions.*

Exercise. Show that the integral

$$\int_T \left\{ \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 \right\} d\sigma$$

if it exists, is invariant of a conformal transformation, and thus can be used to describe an important class of harmonic functions. Write the condition that $\partial u / \partial r$ is summable over the circle in a form which is invariant of conformal transformations.

30. Invariant forms of conclusions. In order to make our theorems general, the conclusions also must be given a form which is invariant of a conformal transformation, i. e., in terms of the Green's function and its conjugate. An application of the theorems of Chapter II gives us at once the following theorems.

THEOREM 1. *Under* (i), *the function* $u(M)$ *has a limit as* M *approaches the frontier* C *along almost all curves* $h = const$; *moreover, if we integrate along curves* $g(AM) = const$, *we have*

$$(2) \qquad \lim_{g=0} \int_{h_1}^{h} u(M)\, dh \text{ exists, } = F(h),$$

for every value of h, *with* $F(h)$ *a function of* h *of limited variation with regular discontinuities. The function* $u(M)$ *of class* (i) *is uniquely determined by this frontier function* $F(h)$.

We may say that $u(M)$ *takes on boundary values for almost all* h *if it behaves as in the first clause of Theorem* 1.

THEOREM 2. *Under* (ii), *the function* $F(h)$ *is absolutely continuous as a function of* h, *and if we write* $f(h) = F'(h)$, *where the latter exists, and* $f(h)$ *arbitrary otherwise, the function* $u(M)$ *takes on the boundary values* $f(h)$, *as* M *approaches* C *along a curve* $h = const.$, *for almost all* h.

Moreover if $f(h)$ *is an arbitrary summable function of* h, *there is one and only one* $u(M)$ *of class* (ii) *which takes on these boundary values. A boundary value* $f(h)$ *will be attained by* $u(M)$ *whenever the condition*

$$f(h) = \frac{d}{dh}\int f(h)\, dh$$

is satisfied.

The second theorem itself yields a solution of the Dirichlet problem for the general simply connected region, and the first theorem gives a solution of a generalized problem of nature similar to that of Dirichlet. In order to make these theorems more useful, however, the functions $f(h)$ and $F(h)$ should be defined not merely in terms of limits of functions of h when g approaches 0, but rather, directly in terms of the boundary points themselves. This problem is primarily

that of the order of boundary points; and we find that for a solution of it, consideration of the so-called "accessible points" of C will suffice.

31. Order of boundary points. A point P of C is *accessible* if it can be joined to an interior point M of T by a simple continuous curve γ. The order of these accessible points is given by means of the behavior of the conformal transformation already mentioned, with relation to the points of C.* We quote as lemmas two theorms of Osgood.†

LEMMA 1. Let T be a simple connected plane region whose boundary consists of more than one point, and let the interior of T be mapped conformally on the interior of a circle S. Let P be an accessible boundary point of T, and let γ be a curve lying within T (except for one extremity) and leading to P. Then the image of γ in S is a curve γ' with a single limiting point P on the circumference of S; so that if a point M approach P along γ its image will approach P' as a limit.

LEMMA 2. Let γ_1 be a second curve of T also leading from a point A of γ to P and meeting γ only in A and P. Let γ_1' be the image of γ_1 in S, P_1' its limiting point on the boundary of S. The necessary and sufficient condition that P_1' coincide with P' is that the simple closed curve $\bar{\gamma}$ consisting of γ and γ_1 may be drawn together continuously to the point P without passing out of T; or in other words, that $\bar{\gamma}$ shall contain in its interior only interior points of T.

In particular, the proof of Osgood's theorem shows that if P is an accessible point there is a definite value of $h(A|P) = h_0(O|P')$ that corresponds to the limiting position of M as M approaches P along γ; moreover P is accessible along that $h_A =$ const. curve or is a limit point of that curve. The same remark also applies to some value $h(B|P)$, and

* The properties of the boundary points are investigated by Osgood, Carathéodory and Courant. [See Hurwitz-Courant, *Funktionentheorie*, Berlin (1922), p. 349.]

† Osgood and Taylor, *Conformal transformations on the boundaries of their regions of definition*, Transactions of the American Mathematical Society, vol. 14 (1913), pp. 277–298.

to accessibility along $h_B =$ const., where B is any other pole; for we might have made a conformal transformation with B going into O. For convenience let us refer to these two h curves from A and B, respectively, determined by P, as λ and λ_1. The curve λ (or λ_1) cannot itself lead to two different accessible positions on C, as an immediate consequence of Lemma 1.

As is indicated in Lemma 2, however, two different h curves from A may belong to the same position P on C. Such positions we describe as *multiple points* and we define each accessible part of the position which belongs to an h curve from A as a separate accessible point P, determined by a specific value of h. So far the identity of a point refers to a single pole of reference A, and the curve λ which belongs to it. But if B is a second pole in T, the preceding paragraph tells us that we can set up a one to one correspondence between curves λ from A and curves λ_1 from B which belong to accessible points; for the relation described is a symmetic one. The curves λ and λ_1 have images in each conformal transformation, λ', λ_1' and λ'', λ_1'' respectively, and λ' and λ_1' lead to the same point P' on the circumference of S while λ'' and λ_1'' lead to the same point P'' on the circumference of S. As we see from Lemma 2, a necessary and sufficient condition that λ and λ_1 correspond if they actually lead to points of C is that the closed curve formed by λ, λ_1 and $h(A|B)$ contain in its interior no points of C. The individuality of an accessible point of C, as now defined, is thus independent of the pole A. Moreover Lemma 2 enables us to decide whether any curve, not necessarily an h curve, which leads to a point C, leads or does not lead to the particular point P if P is accessible along an h curve.*

We now assign to the accessible points of C an order. It will be defined merely as the circular order of the values $-h(A|P)$. The order is thus defined with reference again

* Carathéodory gives an example of an accessible boundary point, of a simply connected region, of non-denumerable multiplicity, Math. Annalen, vol. 73 (1913), p. 363.

to a particular pole A. But this order is independent of the pole which is chosen, or what amounts to the same thing, of the conformal transformation which identifies the order of the points of C with the counter clockwise order of boundary points of the circle. For any conformal transformation of S into T is merely one of the interior of the circle into itself, which does not change the order of points on the circumference, followed by the particular conformal transformation in which $-\theta$ corresponds to the values of $h(A\,P)$. The order of the accessible boundary points as above defined is thus an intrinsic property of the region T.

If we have a family of curved rays γ leading from an arbitrary point A of T to points of C, such that one goes to every accessible point as above defined, and no two have common points in T except at A, the order of the points of C will be the same as the counter clockwise order of the rays from A. In fact, the images of these rays in S make a similar family of rays γ' with the same order about O, and this again is the counter clockwise order of their end points on the circumference of the circle. This is evident if we remember that having drawn one ray γ'_α, any second ray γ'_β forms a cross out in the resulting simply connected region. If these are divided by another pair of rays γ'_μ, γ'_ν, their end points will be similarly divided by the end points of the latter pair of curves.

We have stated that it is sufficient to consider the accessible points of C.* This is so, because we can prove the following theorem:

THEOREM 3. *Almost all the h curves issuing from A lead to accessible points of C.*†

The truth of the theorem follows at once from the lemma

* In fact, as follows from Theorem 3, it would be sufficient to consider the points accessible along curves $h_s =$ const., for some pole A.

† Osgood shows that the set of values of h corresponding to accessible points of C is dense [loc. cit.]. Carathéodory [loc. cit. p. 365] proves the theorem given above by a different method. The method given above is taken from our article on Fundamental Points of Potential Theory, already referred to.

that *almost all the h curves issuing from A are finite in length.* We do not mean of course that their length is bounded for this set of values.

In order to prove this lemma we consider first the part of T contained between two curves $g(A \mid M) = k$ and $g(A \mid M) = k'$, and form the integral

$$\int_\sigma \left\{ \left(\frac{\partial g}{\partial x} \right)^2 + \left(\frac{\partial g}{\partial y} \right)^2 \right\} d\sigma = -\int_s g \frac{\partial g}{\partial n} ds = (k - k')\, 2\pi,$$

where s denotes the complete boundary of σ and n the interior normal. But this integral has the upper bound $2\pi k$ for all k', $0 < k' < k$, and since the integrand is not negative it follows that the squares of the partial derivatives of $g(A \mid M)$ are summable over T when the neighborhood of the pole A is excluded.

Now let ds_g be the element of arc of a curve $h = \text{const.}$ in the direction of increasing g, and let ds_h be similarly defined for h. Then we may write

$$I_h = \int_0^1 ds_g$$

as the length of the portion of the curve $h = \text{const.}$, finite or infinite, outside the contour $g = 1$. Now everywhere in T

$$0 \leqq \frac{dg}{ds_g} \leqq \left(\frac{\partial g}{\partial x} \right)^2 + \left(\frac{\partial g}{\partial y} \right)^2 + 1$$

so that the integral

$$\int_{\sigma'} \frac{dg}{ds_g} d\sigma$$

exists, if σ' is the portion of T outside $g = 1$. But we have

$$\int_{\sigma'} \frac{dg}{ds_g} d\sigma = \int_{(g<1)} \frac{dg}{ds_g} \frac{ds_g}{dg} \frac{ds_h}{dh} dg\, dh = \int_{(g<1)} \frac{ds_h}{dh} dg\, dh$$
$$= \int_{(g<1)} \frac{ds_g}{dg} dg\, dh,$$

and the integral may be written as an iterated integral

$$\int_0^{2\pi} dh \int_0^1 \frac{ds_g}{dg}\, dg = \int_0^{2\pi} I_h\, dh$$

from which follows the summability of I_h.* To say that I_h is summable however is to say that it is finite *almost everywhere.*

This completes the study of the order of the accessible points of C. The inaccessible points may however also be treated by blocking them off in subsets, yielding elements called *Primenden* by Carathéodory, in the memoir to which references have been given. He establishes the remarkable theorem that the *Primenden* (including accessible points among them) may be put into one to one correspondence with the values of h. But this complete order is not necessary for us.

32. Integrals on the boundary and the Dirichlet problems. We shall say that a function $f(P)$ is defined *almost everywhere on the frontier* (or boundary) *of* T if it is defined on accessible points of the frontier which correspond to *allmost all* values of h, and that it is *summable with respect to* h if when defined on the rest of the interval from 0 to 2π in h it becomes a summable function of h. We notice that if $f(P)$ is summable with respect to $h(A, P)$, A being a point of T it is also summable with respect to $h(B, P)$, B being any other point of T. In fact the mapping on S shows that $h(B, P)$ is a continuous function of $h(A, P)$ with continuous derivative, on the set of values of $h(A, P)$ which belong to accessible points of C, and that it can be defined so as to be continuous with its derivative for all values of $h(A, P)$.

If a function $F(P)$ is defined almost everywhere on C, and is of limited variation with respect to h on the set of values of h for which it is defined, it may by Theorem 1, Chap. I, be extended so as to be of limited variation for all values of h in the interval $(0, 2\pi)$. Moreover the property is

* de la Vallée Poussin, *Cours d'Analyse Infinitésimale,* vol. II (1912), p. 121.

independent of the pole A to which the function h refers. We shall therefore speak of $F(P)$ as being of *limited variation on* C. If P is an accessible point of C the quantities $F(P+0)$ and $F(P-0)$ (which refer to the order of points of C) both exist, and if $F(P) = \frac{1}{2}\{F(P+0)+F(P-0)\}$ the discontinuities may be regarded as regular. In fact the function may be extended, as we have seen, so as to have regular discontinuities as a function of h for all h. The function $F(P)$ need not be single valued on C (i. e., periodic as a function of h). We shall consider however merely functions $F(P)$ whose successive multiple values at a generic point of C differ by a single constant value (i. e., $F(h+2\pi) = F(h)+F(2\pi)-F(0)$).

We are now in a position to define the two integrals

$$\text{(E)} \qquad u(M) = \frac{1}{2\pi}\int_C \frac{dh(M, P)}{dh(A, P)}\, dF(P),$$

$$\text{(F)} \qquad u(M) = \frac{1}{2\pi}\int_C f(P)\, dh(M, P),$$

the integration being extended over values of h from 0 to 2π, A being a fixed and M a variable point of T, and P being a point of C. In particular, the integral (E) may, as is easily verified, be formed or defined directly in terms of a Riemann or Cauchy sum on the accessible points of C. Moreover if we have one class of functions on C which are summable with respect to h, we may define the integral (F), directly on C, for further classes of functions by the method of increasing and decreasing sequences of functions.* The integral may also be regarded from the point of view of the general integral of Daniell, which in effect amounts to the same thing. Possible primary classes of functions will be given shortly. The relation of the secondary functions, defined as limits, to the Dirichlet problem is stated in Theorem 6, below.

A conformal transformation which carries T into S and A into the center of S, transforms (E) into the Poisson-Stieltjes

* Cf. Evans, *Fundamental points of potential theory*, p. 322 and 329. Wiener, Trans. Amer. Math. Soc., vol. 25 (1923), p. 307.

integral (A) of Ch. II, and (F) into the Poisson integral (B) of Ch. II, by the corresponding transfer of values, and the inverse transformation performs the reverse transfer; hence we have the following theorems on the generalization of the Dirichlet problem and the problem itself.

THEOREM 4. *A necessary and sufficient condition for the representation* (E) *is given by* (i), *or one of its equivalents. Equation* (2) *holds except for an additive constant for all accessible points of C except the denumerable infinity where $F(P)$ is discontinuous, and for these also if the discontinuities are made regular; moreover if $u(M)$ is of the class* (i) *it is uniquely determined by $F(P)$ and equation* (2).

THEOREM 5. *A necessary and sufficient condition for the representation* (F) *is given by* (ii). *The $u(M)$ takes on the boundary values $f(P)$ almost everywhere on C and is uniquely determined by them, if of the class* (ii). *The function $f(P)$ is arbitrary provided that it is summable with respect to $h(A|P)$, and the boundary value is taken on wherever $f(P)$ is the derivative of its integral with respect to $h(A|P)$ both of these characterizations being independent of the choice of the point A.*

Consider denumerable sequences of not negative functions, for which a limiting function exists. Since in this case, a necessary and sufficient condition for the limit of the integral to be the integral of the limit, is the uniform absolute continuity of the integral of the function of the sequence, we have at once the following theorem on the limits of Dirichlet problems:

THEOREM 6. *Let $f_m(P)$ be a denumerable sequence of functions defined almost everywhere on C and not negative. Let $f_m(P)$ have a limit $f(P)$ almost everywhere on C, and let $f_m(P)$ and $f(P)$ be summable with respect to h and such that $\lim\limits_{m-\infty}\int f_m(P)\,dh = \int f(P)\,dh$. Then, if $u_m(M)$ and $u(M)$ are the harmonic functions of class* (ii) *determined respectively by the boundary values $f_m(P)$ and $f(P)$, we shall have*

$$(3) \qquad \lim u_m(M) = u(M), \qquad M \text{ in } T.$$

COROLLARY. *Equation* (3) *applies if the* $f_m(P)$, *instead of being not negative, form an increasing sequence of summable functions with summable limit.*

In fact the integral of the limit is the limit of the integral, and since the functions $f_m(P)-f_1(P)$ form a not negative sequence the theorem on uniform absolute continuity may again be applied.

It may be opportune to emphasize that in this theorem we are not merely establishing a one-one correspondence between limits of functions on the boundary and limits of functions harmonic in T. Since the limit of the harmonic functions is of class (ii), being given by (F), it follows that it actually takes on almost everywhere the limit of the $f_m(P)$ and is determined by those boundary values — a fact which would not be true if we did not limit the $u_m(M)$ to the class (ii).

33. Special cases of the condition (ii). The continuous boundary value problem. A more usual statement of the Dirichlet problem is solved by the following theorem:

THEOREM 7. *Let* $f(M)$ *be continuous in a region* $\sum$ *which contains* T *and its boundary* C. *There is one and only one function bounded and harmonic in* T *which takes on the values* $f(P)$ *almost everywhere on* C.

In fact $f(P)$ is bounded on C. It is also integrable with respect to h; for almost everywhere on C it is the limit of a continuous function of h, $0 \leqq h \leqq 2\pi$, namely the limit of the function $f(M)$ formed on the analytic curve $g(A, M) = \text{const} = c$ as c approaches zero and M moves out along curves $h = \text{const}$.

If we demand merely that $f(M)$ be continuous in T (instead of $\sum$) and "semi-continuous" on C (that is, the value at an accessible point P of C is the limit of the values of $f(M)$ as M approaches P along every arc in T leading to P), and if $f(M)$ is bounded, then the conclusion of Theorem 7 still applies, for it is sufficient to consider merely points of C which are accessible along curves $h = \text{const}$. This is a more general type of continuous boundary value problem

than the preceding one; in fact $f(P)$ may have different values on the various accessible points P that make up the whole or a part of a multiple point of C.

34. A new continuous boundary value problem. These special problems which we have just been describing discuss the application of a harmonic function to a continuous function as a sort of plaster glued around the edges. An interesting class of continuous boundary value problems may be described directly in terms of the boundary points, without the use of any function $f(M)$ on neighboring points of T, and solved in terms of the methods already developed.

Let P_0 be an accessible boundary point, that is, a simple accessible point or a simple accessible part of a multiple point; let s be a circumference with center P_0 and s_{P_0} the arc-cross-cut of it which divides T into two simply connected parts, of which one contains P_0. Let σ_{P_0} be this part of T. Then we say that $f(P)$, given almost everywhere on C, is continuous at P_0 if given ε we can take the radius r of s sufficiently small so that

$$|f(P)-f(P_0)|<\varepsilon,$$

provided P is any accessible boundary point of T in σ_{P_0}.

We say that $f(P)$ is continuous on the accessible points of the boundary of T if it is continuous at every accessible point of the boundary.

Lemma. A function $f(P)$ continuous at the accessible points of the boundary of T is measurable with respect to $h(A, P)$.

Osgood has shown that to the ends of s_P, namely P_1 and P_2 correspond two values of $h(A, P)$, say h_1 and h_2, which are unequal and unequal to h_P, such that

$$\begin{aligned}|h_1-h_P|&<\eta\\|h_2-h_P|&<\eta\end{aligned}\qquad(\eta \text{ arbitrary}),$$

provided we approach the boundary of T along the two directions of s_P, if the radius of the circle s is taken small enough, equal say to r'. All the intermediate values of h,

$h_1 \leqq h \leqq h_2$, which correspond to curves $h =$ const. which lead to points of C, are such that those curves cut $s_P = s'_P$, and ultimately remain within σ'_P, or reach the points P_1 and P_2, since no two curves $h =$ const. may intersect at an interior point of T. If then, we take any value of $r > r'$ all the above values of h give curves that lead to points of σ_P which are boundary points of T.

In other words, if θ_0 is the value on the unit circle which corresponds to P_0 by the conformal transformation, we can, given ε, find δ (equal to the smaller of two values $|h_1 - h_{P_0}|$, $|h_2 - h_{P_0}|$) such that if $|\theta - \theta_0| < \delta$ we shall have $|f(P) - f(P_0)| < \varepsilon$ for all the corresponding boundary points of T which are accessible along curves $h =$ const. In fact these curves $h =$ const. lead to boundary points of σ_{P_0} which are boundary points of T.

We return now to the lemma to be proved. It suffices to show that the set of values of θ corresponding to accessible points of C for which $f(\theta) > b$, when b is given arbitrarily, is measurable; we denote this set by E_b. Let F be the set of values of θ, for which the corresponding h curve does not lead to an accessible point; F is of measure 0. Now if θ_1 is a value of θ for which $f > b$ there will be, as we have just shown, an open interval ω_1, with center at θ_1, such that

$$CF \cdot \omega_1 < E_b.$$

Hence the set E_b is identical except for a set of measure 0, with the set

$$\sum (CF) \cdot \omega_1 = CF \cdot \sum \omega_1$$

where the intervals ω_1 are formed for all points of E_b. But the sum of an infinity of open sets, denumerable or not, is an open set. Accordingly $\sum \omega_1$ is an open set and therefore measurable. Also CF is measurable, since F is measurable. It follows then that E_b which is, except for a set of zero measure, the product of two measurable sets, is measurable.

From Theorem 5 we have now at once the following one.

THEOREM 8. *If $f(P)$ is continuous on the accessible points of C, and bounded, there is one and only one function $u(M)$,*

harmonic in T, and bounded, which takes on almost everywhere on C the given values $f(P)$. The function $u(M)$ takes on the value $f(P)$ at every accessible point.

In fact, $f(P)$ being measurable with respect to h, and bounded, is summable with respect to h. This proves the first part of the theorem.

In order to prove the second part it is sufficient to show that at any accessible point P_0, $f(P_0)$ is the derivative of its integral with respect to h. Let $\bar{f}(\theta)$ be the corresponding function on the circumference of S, and θ_0 the value which corresponds to P_0. For the moment we define $\bar{f}(\theta)=f(P_0)$ for all the points of the set F of the lemma, which does not affect the value of the integral with respect to h or θ, since F is of measure 0. Hence by taking δ small enough there will be an ε such that

$$f(P_0)-\varepsilon<\bar{f}(\theta)<f(P_0)+\varepsilon, \qquad |\theta-\theta_0|<\delta,$$

and for any set e in this neighborhood it will follow that

$$\begin{aligned}(f(P_0)-\varepsilon)\cdot \text{meas } e &< \int_e \bar{f}(\theta)\, d\theta \\ &= \int f(P)\, dh < (f(P_0)+\varepsilon)\cdot \text{meas } e.\end{aligned}$$

Hence the derivative of the integral of $f(P)$, with respect to h, at P_0 lies between $f(P_0)-\varepsilon$ and $f(P_0)+\varepsilon$. But ε is arbitrarily small.

Exercise. Consider the extension of this theorem when the transformation is conformal at P on the boundary. Hence state a theorem which applies to a region bounded by a finite number of branches of analytic curves.

35. The generalized Neumann problem in the general region. For problems which refer to boundary values of the derivative we make use of the invariants

$$\int_{M_1}^{M_2} \frac{\partial u}{\partial n}\, ds = -\int_{h_1}^{h_2} \frac{\partial u}{\partial g}\, dh,$$

$$\left|\int_{M_1}^{M_2}\left|\frac{\partial u}{\partial n}\right| ds\right| = \left|\int_{h_1}^{h_2}\left|\frac{\partial u}{\partial g}\right| dh\right|,$$

where the integration is extented along curves $g = \text{const.}$, where n denotes the interior normal and ds is (for the purpose of this section) an algebraic magnitude counted positive in the direction opposite to increasing h.

We shall say that $u(M)$ harmonic in T, belongs to the class (j) if its total absolute flux on the level curves $g(A|M) = \text{const.}$ is bounded:

$$\int_{g=\text{const}} \left| \frac{\partial u(M)}{\partial n} \right| ds < N.$$

We shall say that the flux takes on the boundary values given by $F(P)$, where $F(P)$ is periodic, if

$$\lim_{g=0} \int_{M_1}^{M_2} \frac{\partial u}{\partial n} ds = F(P_2) - F(P_1)$$

where the integration is extended along curves $g = \text{const.}$, and M_1 and M_2 approach C respectively along curves $h = \text{const.}$ which belong to accessible points P_1 and P_2. We need not, as a matter of fact, restrict the path of integration in this integral to curves $g = \text{const.}$, since the value of the integral is the same over any other rectifiable path in T joining M_1 and M_2. We must however take $F(P)$ as single valued, since the integral of $\partial u/\partial n$ over a closed curve in T is 0.

The conformal transformation yields at once the following theorem.

THEOREM 9. *Given $F(P)$, single valued and of limited variation on C, there is one and only one function of class (j), harmonic in T, whose flux takes on the boundary values given by $F(P)$ except at the points of discontinuity of $F(P)$.**

* For a detailed study of a more special case of the discontinuous boundary value problems of the first and second kinds, see Lichtenstein, Journal für Math. vols. 141, 142, and 143 (1912–14). He treats also the extension to partial differential equations of elliptic type.

CHAPTER VI

PLANE REGIONS OF FINITE CONNECTIVITY

36. Functions harmonic outside a circle. Consider a circle of radius a and center O, and let $u(r, \theta)$ be harmonic, $r > a$. We say that $u(r, \theta)$ is regular at ∞ if $u(r, \theta)$ is bounded outside a circle of radius R, $R > a$.

If we make a transformation of the plane by inversion, say in a circle of radius 1 and center O, the transferred values of the given function $u(r, \theta)$, regular at ∞, yield a function $U(r', \theta)$ which is harmonic inside the circle of center O and radius $a' = 1/a$, except possibly at O, in the neighborhood of which point the function remains bounded. It is not difficult to show that the function either is harmonic at O also, or else merely has an unnecessary discontinuity, so that it becomes harmonic at O by a proper definition at that point. The fact becomes evident from the following theorem due to Lebesgue.*

LEMMA. A THEOREM OF LEBESGUE. Let $v(r, \theta)$ be harmonic and bounded in the neighborhood of a point O Then it is also harmonic at O or else has an unnecessary discontinuity at that point.

In order to prove this lemma, draw a circle with center O and radius b, small enough to lie entirely within the given neighborhood, and draw a smaller concentric circle of radius ϱ. Let $v_1(r, \theta)$ be the function which is harmonic within the first circle, bounded, and reduces to $v(b, \theta)$ when $r = b$; this function is in fact given by Poisson's integral. The difference, $V(r, \theta) = v(r, \theta) - v_1(r, \theta)$, vanishes as r approaches b and is bounded, say

$$M > V(r, \theta) > m.$$

Within the region between the circles ϱ and b, the function $V(r, \theta)$, being harmonic, satisfies the inequality

$$M \frac{\log b/r}{\log b/\varrho} \geqq V(r, \theta) \geqq m \frac{\log b/r}{\log b/\varrho}.$$

* Comptes Rendus t. 176 (1923), p. 1097.

In fact, otherwise, the difference between $V(r, \theta)$ and one of the other two members of the inequality would have a maximum or a minimum at an interior point of the region.

But now if we hold r fixed in this region, and let ϱ approach zero, both extreme members of the inequality become arbitrarily small. Hence $V(r, \theta) = 0$. Hence $v(r, \theta) = v_1(r, \theta)$, $0 < r < b$, and $v(r, \theta) = v_1(r, \theta)$ at the origin or else can be made harmonic within the circle by defining it that way at the origin.

The function $U(r', \theta)$, to which we return, will consequently be made harmonic within a circle of radius a', and will therefore be given in terms of its values on the circumference of any circle of radius $R' < a'$ by Poisson's integral

$$U(r', \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R'^2 - r'^2)\, U(R', \varphi)}{R'^2 + r'^2 - 2R'r' \cos(\varphi - \theta)} \cdot d\varphi .$$

If therefore we return to the original function $u(r, \theta)$ by inverting back again, we find

$$(1) \qquad u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(r^2 - R^2)\, u(R, \varphi)}{r^2 + R^2 - 2Rr \cos(\varphi - \theta)}\, d\varphi$$

where $R = 1/R' > a$. Thus we have

THEOREM 1. *If $u(r, \theta)$ is harmonic when $r > a$, and regular at ∞, it satisfies the identity* (1) *for $r > R > a$.*

We have also the following results of which the proofs are evident.

COROLLARY. Under the hypotheses of Theorem 1, we have

$$(2) \qquad \begin{cases} \lim\limits_{r=\infty} u(r, \theta) = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} u(R, \varphi)\, d\varphi, \\[2ex] \lim\limits_{r=\infty} r \dfrac{\partial u}{\partial \theta} = \dfrac{R}{\pi} \displaystyle\int_0^{2\pi} u(R, \varphi) \sin(\varphi - \theta)\, d\varphi, \\[2ex] \lim\limits_{r=\infty} r^2 \dfrac{\partial u}{\partial r} = -\dfrac{R}{\pi} \displaystyle\int_0^{2\pi} u(R, \varphi) \cos(\varphi - \theta)\, d\varphi, \end{cases}$$

all uniformly in θ.

It follows that $\lim_{r=\infty} r\frac{\partial u}{\partial x} = 0$, no matter what the direction of x, and uniformly; a property which is included usually in the definition of regularity.

An inversion in the unit circle also gives us the following theorems.

THEOREM 2. *A necessary and sufficient condition that $u(r, \theta)$ be given by a Poisson-Stieltjes integral*

$$(\mathrm{A}') \quad u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2 - a^2)\,dF(\varphi)}{r^2 + a^2 - 2ar\cos(\varphi - \theta)}, \qquad r > a,$$

where $F(\varphi)$ is of limited variation, with $F(2\pi + \varphi) = F(\varphi) + F(2\pi) - F(0)$, is that $u(r, \theta)$ be harmonic for $r > a$, regular at ∞, and satisfy what corresponds to the condition (i) *for the exterior region:*

(i′) $\int_0^{2\pi} |u(r, \theta)|\,d\theta$ *is bounded in an exterior neighborhood of $r = a$.*

If in this theorem we replace (A′) *by the Poisson integral*

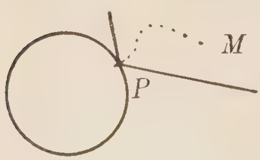
$$(\mathrm{B}') \quad u(r, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2 - a^2)f(\varphi)\,d\varphi}{r^2 + a^2 - 2ar\cos(\varphi - \theta)}, \qquad r > a,$$

with $f(\varphi)$ summable, we replace (i′) *by*

(ii′) $\int_0^{\theta} u(r, \theta)\,d\theta$ *has uniform absolute continuity in an exterior neighborhood of $r = a$.*

A necessary and sufficient condition for (A′), as in Theorem 2, is that $u(r, \theta)$ be the difference of two not negative functions, harmonic outside the circle and regular at infinity.

If $u(r, \theta)$ is given by (A′) it takes on the boundary values $F'(\varphi)$ almost everywhere, and if given by (B′) the boundary values $f(\varphi)$ almost everywhere, provided $M = (r, \theta)$ approaches $P = (a, \varphi)$ in the wide sense, that is to say, in any way so that it remains in the angle formed by two rays directed outward from P, neither ray being tangent to the circle. The other properties of the integrals discussed in Chap. II have their obvious analogs



for (A′) and (B′) and it is unnecessary to state them in detail.

EXERCISE 1. Obtain values for u_∞, $\left[r\dfrac{\partial u}{\partial \theta}\right]_\infty$ and $\left[r^2\dfrac{\partial u}{\partial r}\right]_\infty$ when $u(r, \theta)$ is given by (A′).

EXERCISE 2. Show that a necessary and sufficient condition that

$$v(r, \theta) = \frac{a}{2\pi}\int_0^{2\pi} \log(a^2+r^2-2ar\cos(\varphi-\theta))\,dF(\varphi)+A, \quad r>a,$$

where A is a constant, and $F(\varphi)$ is of limited variation, periodic and with regular discontinuities, is that $v(r, \theta)$ be harmonic outside the circle of radius a and regular at ∞, and that the total absolute flux

$$\int_0^{2\pi} \left|\frac{\partial v}{\partial r}\right| d\theta$$

be bounded for the exterior neighborhood of $r = a$.

Show that $\lim\limits_{r\to\infty} v(r, \theta) = A$ uniformly, that $\lim\limits_{r=a}\displaystyle\int_0^\theta \frac{\partial v}{\partial r}\,d\theta = F(\theta)-F(0)$, and that $\lim\limits_{r-a} \partial v/\partial r = F'(\varphi)$ almost everywhere, in the wide sense.

EXERCISE 3. Describe the special case when $F(\varphi)$ is absolutely continuous.

37. The multiply connected region bounded by $n+1$ distinct circles. Let S be an open region of finite connectivity whose boundary consists of non-intersecting circles, an external one s_0 and n internal ones $s_1, s_2, \cdots, s_n$. Let M be a point of S with polar coördinates r_i, θ_i referred to the center of the circle s_i of radius a_i, and let $u(M)$ be a function harmonic in S. Such a function may be written as the sum of $n+1$ functions $u_i(M)$,

$$(3) \qquad u(M) = u_0(M) + \sum_1^n u_i(M),$$

one of them, namely $u_0(M)$, being harmonic in the open region inside s_0, and each of the others being given by a formula

(3′) $$u_i = w_i + m_i \log r_i, \qquad i = 1, 2, \cdots, n,$$

where w_i is harmonic outside s_i, is regular at ∞ and vanishes there.* We wish to find necessary and sufficient conditions on $u(M)$ in order that $u_0, w_1, \cdots, w_n$ may be expressed by Poisson-Stieltjes integrals of the form (A) and (A′).

But a function $m_i \log r_i$ is continuous on and in the neighborhood of any contour s_j, and a function u_i is continuous on and in the neighborhood of any contour s_j, $i \neq j$, so that we have at once the theorem:

THEOREM 3. *A necessary and sufficient condition that $u(M)$ be given by* (3), (3′) *where the u_0 is given by* (A)

(A) $$u_0(r_0, \theta_0) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(a_0^2 - r_0^2)\, dF_0(\varphi)}{a_0^2 + r_0^2 - 2a_0 r_0 \cos(\varphi - \theta_0)},$$

with $F_0(\varphi)$ of limited variation and $F_0(\varphi + 2\pi) = F_0(\varphi) + F_0(2\pi) - F_0(0)$, and the w_i are given by (A′)

(A′) $$w_i(r_i, \theta_i) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(r_i^2 - a_i^2)\, dF_i(\varphi)}{a_i^2 + r_i^2 - 2a_i r_i \cos(\varphi - \theta_i)}$$

with $F_i(\varphi)$ of limited variation and periodic, with period 2π, is that $u(M)$ be harmonic in the open region S and

(4) $$\int_0^{2\pi} |u(M)|\, d\theta_i < K, \qquad K \text{ constant},$$

when the integration is extended over all circles in S concentric with a boundary circle and in its neighborhood.

COROLLARY 1. A necessary and sufficient condition that u_0 or a particular $w_i(M)$ should reduce to the corresponding integral (B) or (B′) is that, in addition, the absolute continuity of $\int_0^{\theta} u(r_i, \theta_i)\, d\theta_i$ be uniform, when the integration is extended over all circles in S concentric with the particular s_i and in its neighborhood.

EXERCISE. Show that a necessary and sufficient condition that u_0 or a particular w_i should be given by the corresponding integral (C) or (C′) is that in addition to the

* Osgood, *Funktionentheorie*, Leipzig (1912), p. 643.

hypothesis of Theorem 3 the integral $\int_0^{2\pi} \left| \frac{\partial u}{\partial r_i} \right| d\theta_i$ should be bounded for all circles in S concentric with the particular boundary circle s_i and in its neighborhood. This condition of course implies the condition (ii) or (ii′) for the particular circle.

Corollary 2. If $u(r, \theta)$ is harmonic in S and is either bounded or such that $\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\}$ is summable over S, then all the boundary integrals reduce to Poisson integrals.

We shall now prove the following theorem, of which the importance from the physical point of view hardly needs emphasis.

Theorem 4. *A necessary and sufficient condition for the situation of Theorem 3 is that $u(M)$ be the difference of two not negative functions, each harmonic in the open region S.*

The condition is obviously necessary, since the $F_i(\varphi)$ are of limited variation. In order to prove the sufficiency we establish first two simple lemmas.

Lemma 1. For the purpose of this lemma let $\bar{S}$ be a simply connected open region, bounded internally or externally by a circle $\bar{s}$ of radius $\bar{a}$, and let $u(M)$ be the difference of two not negative harmonic functions $u'(M)$, $u''(M)$ in an annular neighborhood in $\bar{S}$ of the boundary $\bar{s}$. Then $\int |u| \, d\theta$ is bounded for the circles of that neighborhood concentric with $\bar{s}$.

For convenience let $\bar{S}$ be an interior region. Let s' of radius r' be a fixed circle in the annular neighborhood and let there be a circle of radius r, $r' \leqq r < a$, all these circles having the same center O. The lemma follows immediately from Bôcher's device, evaluating the area integral of $(1/r)\, \partial u/\partial r$. In fact by integrating this expression in u' over an annular region, and equating its two evaluations as an iterated integral, we have the equation

$$(5) \qquad \left(\int_0^{2\pi} u' \, d\theta\right)_r = \left(\int_0^{2\pi} u' \, d\theta\right)_{r'} + \alpha_1 \log r - \alpha_1 \log r'$$

where α_1 is a constant. Hence

$$\int_0^{2\pi} u' d\theta = \int_0^{2\pi} |u'|\, d\theta$$

is bounded. Similarly $\int |u''|\, d\theta$ is bounded, and therefore $\int |u|\, d\theta$ is bounded.

Lemma 2. In the open region S let $u(M)$ be the difference of two functions $u'(M)$, $u''(M)$, harmonic and not negative in S. Then $u(M)$ is the sum of $n+1$ functions $u_i(M)$, each harmonic except for possible logarithmic singularities at ∞, in the simply connected open region S_i bounded by s_i, and each satisfying in its region S_i the conditions of Lemma 1.

The first part of the lemma is the theorem already expressed equations (3), (3′). The last part follows immediately by writing, in the neighborhood of s_i, $u_i(M)$ as the difference $u(M) - \bar{u}(M)$ where

$$\bar{u}(M) = \sum_j' u_j(M),$$

the summation not including $j = i$.

It can now be shown that the condition of Theorem 4 is sufficient. In fact, this condition is the hypothesis of Lemma 2, and therefore by Lemma 1 the quantity $\int |u_i|\, d\theta_i$ is bounded for circles concentric with s_i and in its neighborhood. The same is true for $\int |u_j|\, d\theta_i$, $j \neq i$, since $u_j(M)$ is harmonic on s_i itself if j is unequal to i. But

$$\int_0^{2\pi} |u|\, d\theta_i \leqq \sum_{j=0}^{n} \int_0^{2\pi} |u_j|\, d\theta_i.$$

Hence the condition (4) of Theorem 3 is satisfied.

38. Representation in terms of the Green's function. The class of functions so far considered may be given a representation directly in terms of the boundary values, rather than in terms of the functions $F_i(\theta)$. But for this we need

one or two preliminary facts about the Green's function for S.*

LEMMA 1. Let n denote the interior normal at the boundary of S. If $g(Q, M)$ is the Green's function for S with pole at Q, there will be two positive numbers l, L, with $0<l<L$, such that the relation

$$l<\frac{\partial g}{\partial n}<L, \tag{6}$$

is satisfied for every point P of the boundary; i. e., of $s_0, s_1, \cdots, s_n$.

The fact is established immediately for any particular boundary s_i (whence the general statement follows) by drawing a circle s' concentric with s_i and lying between it and Q or any other boundary circle s_j. On s', $g(Q, M)$ neither vanishes nor becomes infinite. The function $g(Q, M)$ will then be contained between two particular harmonic functions $\underline{u}(M)$ and $\bar{u}(M)$ both of which vanish on s_i and are given constant values on s', so that they are both linear functions of $\log r_i$,

$$\underline{u}(M) = k\frac{\log r_i - \log a_i}{\log r' - \log a_i}, \qquad \bar{u}(M) = K\frac{\log r_i - \log a_i}{\log r' - \log a_i},$$

where k is >0 but $<g(Q, M)$ on s', and K is finite but $>g(Q, M)$ on s'. But the normal derivative of any function which vanishes on s_i is merely the limit of its value on s' divided by $r_i - a_i$ as r_i tends to a_i; hence direct calculation of the derivatives of the three functions $\underline{u}$, $\bar{u}$ and g establishes the result.

Consider now one of the internal boundaries s_i. From what we have just proved, for a sufficiently small value of x the complete curve c' given by $g(Q, M) = x$ contains, as a part, a separate closed branch c_i' which encloses a single s_i in its interior. Let us consider in particular this interior boundary and the region S' between s_i and c_i'. Analogous considerations will apply of course to s_0.

* For the existence of the Green's function for S see e. g., Lichtenstein, *Neuere Entwicklung der Potentialtheorie*, Encyklopädie der mathematischen Wissenschaften, Band II, Teil 3, Leipzig (1921) § 17, § 20. See also Chap. VII below.

The points on s_i and on c_i' correspond to the same values of h, say the range of values $h' \leqq h < h''$, where $h = h(Q, M)$ is the (non-uniform) harmonic function conjugate to $g(Q, M)$. If we map conformally the region S' on an annular region S'' by the transformation

$$w = e^{-(g+hi)\frac{2\pi}{h''-h'}},$$

the curves $g =$ const. become concentric circles in S'' between $r = e^{-x}$, corresponding to c_i', and $r = 1$, corresponding to s_i, the map being conformal within and on the boundaries.

Assume now that in S' the integral $\int_{h'}^{h''} |u(M)|\, dh(Q, M)$ extended along a curve $g(Q, M) = y$ is bounded as y approaches zero. That implies that in S'' the integral $\int_0^{2\pi} |u|\, d\varphi$ is bounded $e^{-x} \leqq r < 1$, and hence, as a property of the integral formulae of Theorem 3, that $\lim\limits_{r=1} \int_0^\theta u(r, \theta)\, d\theta$ exists for all θ. In S' it follows therefore that the integral $\int_{h_1}^{h_2} u(M)\, dh(Q, M)$, extended along $g = y$, has a limit $U(h_1, h_2)$ as y approaches zero for all h_1, h_2 in the given interval $h' \leqq h_1 \leqq h_2 \leqq h''$.

For the purpose of discussing the function U it is perhaps simpler to avoid using the Riemann surface for the non-uniform function h, and make it single valued, by means of cuts, from 0 to 2π. It may be then that for a given boundary circle the corresponding values of h will not form a complete interval, but rather a finite number of distinct intervals (h_1', h_1''), $\cdots$, (h_p', h_p''). We shall speak of the end points of these intervals as cut-points of the boundary. In this case we may use the mapping function

$$w = e^{-2\pi \frac{g+i\left\{h-h_i'+\sum\limits_{1}^{i-1}{}_r(h_r''-h_r')\right\}}{\sum\limits_{1}^{p}{}_r(h_r''-h_r')}}, \qquad h_i' \leqq h < h_i''.$$

By thus defining $U(h_1, h_2)$ on successive portions of the complete boundary of S, we see that if $\int_0^{2\pi} |u|\, dh$ is bounded the quantity

$$(6) \qquad \lim_{y=0} \int_H^h u(M)\, dh(Q, M) = U(P)$$

exists, and may be used to define a function $U(P)$ on the complete boundary which is single valued as a function of h, $0 \leqq h \leqq 2\pi$; it is also single valued as a function of position P on the boundary except for the one point $P_0 = P_{2\pi}$. It is a function of limited variation as a function of h or of any continuous monotonic function of h, such as the arc length, properly defined, along the complete boundary. Moreover its continuities are regular except at the cut-points h_i', h_i'', and the direction of increase of h along the boundary is, except at the cut-points, the same as for the h-functions of the corresponding simply connected regions S_i; the latter point is an immediate consequence of Lemma 1. In fact, as follows from Lemma 1, the branch points of the analytic function $g + hi$ cannot lie on the boundary of S.

The derivatives of h, considered as a function on the Riemann surface for $g + hi$, with respect to r_i and θ_i are single valued and continuous on and in the neighborhood of the boundary of S; for the analytic function $g + hi$, whose various branches differ merely by constant values, may be extended analytically across any boundary circle.* Moreover the quantity

$$\frac{\partial h}{\partial \theta_i} = -\frac{\partial g}{\partial n}$$

does not vanish on the boundary. We wish to consider the quantity

$$\frac{dh(y\,|\,A, M)}{dh(y\,|\,Q, M)} = \frac{\partial h(y\,|\,A, M)}{\partial s_M} \Big/ \frac{\partial h(y\,|\,Q, M)}{\partial s_M},$$

which represents the rate of change of the h-function with pole A for the curve $g(Q, M) = y$ with respect to the h-function with pole Q for the same curve, along that curve. In particular, $g(y\,|\,Q, M) = g(0\,|\,Q, M) - y$ and $h(y\,|\,Q, M) = h(0\,|\,Q, M)$, whence the denominator fraction of the right hand member of

* Osgood, *Funktionentheorie*, Leipzig (1912), p. 672.

the above equation is continuous and different from zero, as y approaches zero and M approaches a point P of the boundary of S.

Consider now the numerator. We can take $y > 0$ near enough to 0 so that the function $g(y'|A, M)$ for $y' \leqq y$ may be extended harmonically over a region which contains in its interior the whole region defined by $g(Q, M) = -Y$, where Y is some positive number, sufficiently small. For this purpose consider the region S'', which we have already used, associated say with s_0. If x_0 is sufficiently small the image of the point A will not lie in S''. Since the curves $g(Q, M) = y$ are transformed into the circles $r = e^{-y}$ and the transform $\bar{g}(M)$ of the function $g(y|A, M)$ vanishes when M is on the curve of index y, the function $\bar{g}(M)$ may be extended harmonically through a region which includes at least the circle inverse to $r = e^{-x_0}$ in the circle $r = e^{-y}$; that is, the circle of radius $e^{x_0 - 2y}$.* Hence if $Y_0 < x_0$, we may choose $y = y_0$ sufficiently small so that $x - 2y_0 > Y_0$; moreover the transform of the function $g(Q, M)$ is itself harmonically extensible through a region which contains the same circle of radius e^{Y_0}. In other words, if $y' \leqq y_0$ the region of harmonic extension of $g(y'|A, M)$ and $g(Q, M)$ in the neighborhood of s_0 includes the points for which $g(Q, M) \geqq -Y_0$.

If we proceed similarly with the other portions of the boundary $s_1, \cdots, s_n$, and define similarly other pairs of numbers (y_i, Y_i), we may choose y as the smallest of the numbers $y_0, \cdots, y_n$ and Y as the smallest of the numbers $Y_0, \cdots, Y_n$. But $g(Q, M) = 0$ consists of $n + 1$ circles; and therefore $g(Q, M) \geqq -Y$ defines a region which includes all these circles in its interior, for $g(Q, M)$ is harmonically extensible across these circles. In particular Y may be taken near enough to zero so that the curve $g(Q, M) = -Y$ consists of $n + 1$ analytic closed curves.

The function $g(y'|A, M) - g(y''|A, M)$ is harmonic at A, in particular, and therefore an application of Green's theorem to the region bounded by $g(Q, M) = -Y$ yields the equation

* Osgood, *loc. cit.*

$$2\pi\{g(y'|A, M)-g(y''|A, M)\}$$
$$=\int_{(-Y)}\{g(y'|A, P')-g(y''|A, P')\}\frac{\partial g(-Y|P', M)}{\partial n_{P'}}ds_{P'},$$

and, by differentiation, the equation

$$2\pi\left\{\frac{\partial g(y'|A, M)}{\partial \mu_M}-\frac{\partial g(y''|A, M)}{\partial \mu_M}\right\}$$
$$=\int_{(-Y)}\{g(y'|A, P')-g(y''|A, P')\}\frac{\partial^2 g(-Y|P', M)}{\partial n_{P'}\,\partial \mu_M}ds_{P'},$$

where μ_M is an arbitrary fixed direction at M, M is excluded from a given arbitrary neighborhood of A but is otherwiese arbitrary in S or on its boundary, and Y satisfies the conditions of the previous paragraph. But $g(Y|P', M)$ and all its derivatives are bounded, since P' is on $g(Q, P')=-Y$, and M is in S or on its boundary; moreover $g(y'|A, P')-g(y''|A, P')$ is of one sign. Hence we may write

$$2\pi\left|\frac{\partial g(y'|A, M)}{\partial \mu_M}-\frac{\partial g(y''|A, M)}{\partial \mu_M}\right|$$
$$\leqq K\int_{(-Y)}|g(y'|A, P')-g(y''|A, P')|\,ds_{P'}.$$

The integrand of the right hand member is bounded for all y', y'' which are $\leqq y$ and $\geqq 0$, and it approaches 0 as $y'-y''$ tends to 0. Hence

$$\lim_{y''=y'}\frac{\partial g(y''|A, M)}{\partial \mu_M}=\frac{\partial g(y'|A, M)}{\partial \mu_M}.$$

and uniformly, in M and in μ_M.

Let now M' be a second point of the same sort as M and $\nu_{M'}$ an arbitrary direction at M'. We have, holding y' fast,

$$\lim_{\substack{M=M'\\ \mu_M=\nu_{M'}}}\frac{\partial g(y'|A, M)}{\partial \mu_M}=\frac{\partial g(y'|A, M')}{\partial \nu_M},$$

where the limit is uniform for all M' and all $\nu_{M'}$. Hence

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$$\lim_{\substack{y'' = y' \\ M = M' \\ \mu_M = \nu_{M'}}} \frac{\partial g(y''|A, M)}{\partial \mu_M} = \frac{\partial g(y'|A, M')}{\partial \nu_{M'}},$$

uniformly in M' and $\nu_{M'}$.

In particular

$$\lim_{\substack{y''=0 \\ M=P}} \frac{\partial g(y''|A, M)}{\partial n_M} = \frac{\partial g(0|A, P)}{\partial n_P},$$

uniformly in P, where M is on $g(Q, M) = y''$ and P is on $g(Q, P) = 0$, and therefore

$$\lim_{\substack{y''=0 \\ M=P}} \frac{\partial h(y''|A, M)}{\partial s_M} = \frac{\partial h(0|A, P)}{\partial s_P}, \tag{7}$$

uniformly in P. We have, therefore, returning to the ratio

$$\frac{dh(0|A, P)}{dh(0|Q, P)} = \frac{dh(A, P)}{dh(Q, P)},$$

the following proposition.

LEMMA 2. Let Q and A be two points of S and let $h(y|A, M)$ be the h-function with pole A for the curve $g(Q, M) = y > 0$. If the $h(y|A, M)$ and $h(y|Q, M) = h(0|Q, M) = h(Q, M)$ are considered on the Riemann surfaces of their respective functions $g + hi$, the quantity

$$\frac{dh(y|A, M)}{dh(y|Q, M)} = \frac{dh(y|A, M)}{dh(Q, M)},$$

which represents the rate of change along the curve $g(Q, M) = y$, is nevertheless a single-valued, continuous function of M in a neighborhood of the boundary of S. Moreover

$$\lim_{\substack{y=0 \\ M=P}} \frac{dh(y|A, M)}{dh(Q, M)} = \frac{dh(A, P)}{dh(Q, P)}$$

uniformly in P, where P is on the boundary of S, and this latter expression is positive for all P.

We are now in position to prove a fundamental theorem.

THEOREM 5. *Let*

$$I(y, h) = \int_0^h u(M)\, dh(Q, M), \quad T(y, h) = \int_0^h |u(M)|\, dh(Q, M)$$

represent integrals along the curve $g(Q, M) = y$ with respect to $h(Q, M)$ made single-valued by means of cuts. A necessary and sufficient condition that $u(r, \theta)$ be the difference of two not negative functions, each harmonic in S is that $u(r, \theta)$ be harmonic in S and that $T(y, 2\pi)$ remain bounded as y approaches 0; *that is, that $I(y, h)$ be of uniformly limited variation as a function of h, as y approaches* 0.

An equivalent theorem may easily be stated in terms of the Riemann surface for $g + hi$, but the present form is perhaps more closely connected with the physical concept of the curves $h =$ const. as lines of flow.

Suppose then that u is the difference $u' - u''$ of two not negative functions, harmonic in S. We have for u' the equation*

$$2\pi u'(Q) = \int_0^{2\pi} u'(M)\, dh(Q, M) = \int_0^{2\pi} |u'(M)|\, dh(Q, M),$$

when the integration is extended over a complete curve $g(Q, M) = y$, with a similar equation for u''. Hence

$$\int_0^{2\pi} |u(M)|\, dh(Q, M) \leqq 2\pi \{u'(Q) + u''(Q)\},$$

and the condition is satisfied.

Suppose now that the condition is satisfied. We shall show first that for any point A of S the function $u(A)$ is given by a formula

$$u(A) = \frac{1}{2\pi} \int_0^{2\pi} \frac{dh(A, P)}{dh(Q, P)}\, dU(P), \tag{8}$$

where $U(P)$ is a function of limited variation on the boundary, $0 \leqq h(Q, P) \leqq 2\pi$.

* This is merely the familiar formula

$$2\pi u'(Q) = \int_{[g=y]} u'(M) \frac{\partial g(Q, M)}{\partial n}\, ds.$$

In order to verify this equation we notice that the familiar equation

$$u(A) = \frac{1}{2\pi}\int_{(g=y)} u(M)\,\frac{\partial g(y\,|\,A, M)}{\partial n_M}\,ds_M$$
$$= \frac{1}{2\pi}\int_{(g=y)} u(M)\,\frac{\partial h(y\,|\,A, M)}{\partial s_M}\,ds_M$$

may, by means of (10), Chap. I, be written in the form

$$\text{(9)}\qquad u(A) = \frac{1}{2\pi}\int_{(g=y)} \frac{dh(y\,|\,A, M)}{dh(Q, M)}\,d\int_0^{h_M} u(M')\,dh(Q, M')$$
$$= \frac{1}{2\pi}\int_{(g=y)} \frac{dh(y\,|\,A, M)}{dh(Q, M)}\,dI(y, h).$$

By hypothesis $I(y, h)$ is of limited variation uniformly in y, and, as we have shown, has a limit $U(P)$ for every value of h, $U(P)$ being a function of limited variation. Moreover, by Lemma 2, the continuous function $dh(y|A, M)/dh\,(Q, M)$ approaches $dh(A, P)/dh(Q, P)$ uniformly, as a function of $h(Q, P)$, $0 \leqq h(Q, P) \leqq 2\pi$. Hence we may apply the Helly-Bray theorem [Theorem 3, Chap. I], which supplies at once the desired equation (8).

But by Lemma 2, $dh(A, P)/dh(Q, P) > 0$, and $U(P)$ being of limited variation, as a function of $h(Q, P)$, is the difference of two non-decreasing functions. Hence $U(A)$ is the difference of the two corresponding not negative functions, harmonic in S. Thus the theorem is proved.

Corollary 1. A necessary and sufficient condition that $u(A)$ be given by (8), where $U(P)$ is of limited variation, $0 \leqq h(Q, P) \leqq 2\pi$, and Q is an arbitrary fixed point of S is that $u(A)$ belong to the class of functions considered in Theorems 3, 4 and 5. This equation furnishes the solution of the generalized Dirichlet problem for S.

Corollary 2. If the condition that $T(y, h)$ be bounded holds with respect to a given pole Q, it holds for any other pole Q'.

Exercise 1. State Theorem 5 in terms of the Riemann surfaces, and describe the discontinuities of the limiting function $U(P)$ on the surface belonging with $h(Q, P)$.

EXERCISE 2. Obtain a formula for the boundary values of $U(A)$ as A approaches P along a curve $h(Q, P) = \text{const.}$, and rewrite (8) for the case that $U(P)$ is absolutely continuous as a function of $h(Q, P)$.

39. Boundary integrals and Stieltjes integral equations. So far we have not established any relation between the boundary function $U(P)$, of limited variation, and the $n+1$ boundary functions of limited variation $U_i(\theta_i)$

$$U_i(\theta_i) = \lim_{r_i = a_i} \int_0^{\theta_i} u(r_i, \theta_i)\, d\theta_i, \qquad i = 0, 1, \cdots, n, \tag{10}$$

and although we know that the former function is arbitrary, we have not established the same fact for the latter functions, nor do we know that for a given harmonic function of the general class that we have been describing the functions $F_i(\theta_i)$ of Art. 37 are essentially determined. This subject will be considered in the present section. We first solve a simple type of linear Stieltjes integral equation.

THEOREM 6. *Let $U(x)$ be of limited variation, $0 \leq x \leq X$, and $K(x)$ continuous in the same closed interval, and positive. The equation*

$$U(x) - U(0) = \int_0^x K(y)\, dV(y) \tag{11}$$

has the solution

$$V(y) - V(0) = \int_0^y \frac{1}{K(x)}\, dU(x) \tag{12}$$

in which $V(0)$ is arbitrary and $V(y)$ a function of limited variation; and this is the only solution in which $V(y)$ is a function of limited variation.

In fact, if $V(y)$ is of limited variation it is the difference of two non-decreasing functions, hence by means of (11) the function $U(x)$ is also the difference of two non-decreasing functions, and is of limited variation. Consequently, if $V(y)$ is of limited variation, we may form the integral

$$I(y) = \int_0^y \frac{1}{K(x)}\, dU(x) = \int_0^y \frac{1}{K(x)}\, d\int_0^x K(\alpha)\, dV(\alpha)$$

and we shall have

$$\left| I(y) - \sum_0^{n-1}{}_i \frac{1}{K(x_i)} \int_{x_i}^{x_{i+1}} K(\alpha)\, dV(\alpha) \right| \leqq \omega_\delta \left(\frac{1}{K} \right) C_1 .$$

where $x_0 = 0,\ x_1, \cdots, x_n = y$ are points which divide $(0, y)$ into n subintervals each of length $\leqq \delta$, $\omega_\delta \left(\frac{1}{K} \right)$ is the maximum oscillation of the reciprocal of $K(x)$ in the interval $(0, X)$ and C_1 is a constant, independent of y. But

$$\left| \int_{x_i}^{x_{i+1}} K(\alpha)\, dV(\alpha) - K(x_i) \{ V(x_{i+1}) - V(x_i) \} \right|$$
$$\leqq \omega_\delta(K) \{ t_V(x_{i+1}) - t_V(x_i) \}$$

where $t_V(x)$ is the total variation function of $V(x)$. Therefore

$$\left| \sum_0^{n-1}{}_i \frac{1}{K(x_i)} \int_{x_i}^{x_{i+1}} K(\alpha)\, dV(\alpha) - \sum_{i\ 0}^{n-1} \{ V(x_{i+1}) - V(x_i) \} \right| \leqq \omega_\delta(K)\, C_2 .$$

where C_2 is a constant, $\leqq t(X)/\text{min. } K(x)$. Accordingly,

$$| I(y) - \{ V(y) - V(0) \} | \leqq \omega_\delta(K)\, C_2 + \omega_\delta \left(\frac{1}{K} \right) C_1 ,$$

and, since δ is arbitrary,

$$I(y) = V(y) - V(0)$$

Similarly if $U(x)$ is of limited variation, and $V(y)$ is given by (12) it follows that $U(x)$ is given by (11). Hence the theorem is proved.

Corollary. If the relation between $U(x)$ and $V(y)$ is given by (11), and $\varphi_U,\ \varphi_V,\ \psi_U,\ \psi_V$ are used to denote the positive and negative variation functions of U and V, then

$$(13) \quad \varphi_U(x) = \int_0^x K(y)\, d\varphi_V(y), \quad \psi_U(x) = \int_0^x K(y)\, d\psi_V(y)$$

and these integrals may be inverted by Theorem 6.

The direct proof of (13) is more tedious than difficult. It follows however almost immediately from Theorem 6. In fact, if we write

$$\bar{\varphi}_V(y) = \int_0^y \frac{1}{K(x)}\, d\varphi_U(x), \quad \bar{\psi}_V(y) = \int_0^x \frac{1}{K(x)}\, d\psi_U(x)$$

we shall have $\bar{\varphi}_V(y)$ a non-decreasing and $\bar{\psi}_V(y)$ a non-increasing function, and

$$V(y) - V(0) = \bar{\varphi}_V(y) + \bar{\psi}_V(y).$$

Hence

$$\bar{\varphi}_V(y) \geqq \varphi_V(y), \quad \bar{\psi}_V(y) \leqq \psi_V(y),$$

since $\varphi_V(y)$ is the least upper bound and $\psi_V(y)$ is the greatest lower bound for their respective generating sums. Also the functions

$$\bar{\varphi}_U(x) = \int_0^x K(y)\, d\varphi_V(y), \quad \bar{\psi}_U(x) = \int_0^x K(y)\, d\psi_V(y)$$

satisfy the inequalities

$$\bar{\varphi}_U(x) \geqq \varphi_U(x), \qquad \bar{\psi}_U(x) \leqq \psi_U(x).$$

But since $\bar{\varphi}_V \geqq \varphi_V$, $\bar{\psi}_V \leqq \psi_V$, we have

$$\int_0^x K(y)\, d\bar{\varphi}_V(y) \geqq \int_0^x K(y)\, d\varphi_V(y)$$

$$\int_0^x K(y)\, d\bar{\psi}_V(y) \leqq \int_0^x K(y)\, d\psi_V(y),$$

so that, by Theorem 6,

$$\varphi_U(x) = \int_0^x K(y)\, d\bar{\varphi}_V(y) \geqq \bar{\varphi}_U(x), \quad \psi_U(x) \leqq \bar{\psi}_U(x)$$

Hence the equality sign alone is possible and the Corollary is proved.

We may now, utilizing these results, find the desired relations between the functions $U_i(\theta_i)$ given by (10) and the function $U(P) = U(h)$ given by (6). We consider

$$I_{0\theta_i} = \int_{\substack{0 \\ (r)}}^{\theta_i} u(M)\, d\theta = \frac{1}{2\pi} \int_{\substack{0 \\ (r)}}^{\theta_i} d\theta_M \int_{(g=0)} \frac{d\,h(M, P)}{d\,h(Q, P)}\, dU(P)$$

where (r) indicates integration over the circle of radius $r_i = r$, concentric with s_i, and $(g = 0)$ indicates integration

over the complete boundary of S. Since M is not on this boundary we may invert the order of integration and write*

$$I_{0\theta_i} = \frac{1}{2\pi}\int_{(g=0)} dU(P)\int_{\substack{0\\(r)}}^{\theta_i}\left[\frac{dh(M,P)}{dh(Q,P)}\right]_{g=0} d\theta_M$$

$$= \frac{1}{2\pi}\int_{(g=0)} \frac{dU(P)}{\frac{\partial g(Q,P)}{\partial n_P}}\int_{\substack{0\\(r)}}^{\theta_i}\frac{\partial g(M,P)}{\partial n_P}\,d\theta_M$$

$$= \frac{2\pi r_M}{1}\int_{(g=0)} \frac{dU(P)}{\frac{\partial g(Q,P)}{\partial n_P}}\int_{\substack{0\\(r)}}^{s}\frac{\partial g(M,P)}{\partial n_P}\,ds_M.$$

where s is the absolute arc length corresponding to the value θ_i.

Now $g(M,P)$ is harmonic, $M \neq P$, even if M and P are points on the boundary, and can therefore be extended across the boundary s_i as a function of M or P when M and P are isolated from each other. Hence $\partial g(M,P)/\partial n_P$ approaches zero uniformly if $|\theta_P - \theta_M|$ remains different from zero, $\geqq \varepsilon$. But $g(M,P)$ minus the Green's function for the circle s_i is harmonic in the neighborhood of the boundary s_i if $M \neq P$, and remains bounded in the neighborhood of $M = P$, even when M and P are points of s_i. Hence

$$\left|\frac{\partial g(M,P)}{\partial n_P} - \frac{1}{a_i}\,\frac{r_M^2 - a_i^2}{a_i^2 + r_M^2 - 2a_i r_M \cos(\varphi_P - \theta_M)}\right|$$

remains bounded, say $< K$, since by Lebesgue's theorem the above difference $g - g_i$ is harmonic as a function of either point, except for unnecessary discontinuities. The following equations may therefore be verified, $i = 1, \cdots, n$, and analogous ones if $i = 0$.

$$(14)\quad \lim_{r=a_i}\int_{\substack{0\\(r)}}^{s}\frac{\partial g(M,P)}{\partial n_P}\,ds_M = 2\pi,\qquad 0 < \varphi_P < \theta_i,$$

$$= 0,\qquad \theta_i < \varphi_P < 2\pi,$$

$$= \pi,\qquad \varphi_P = 0 \text{ or } \varphi_P = \theta_i.$$

* On integration under the integral sign in a Stieltjes integral, see for instance, Evans, *Problems of potential theory,* Rice Institute Pamphlets, vol. 7 (1920), p. 258.

In fact, to take the case where $\varphi_P = \theta_i$, we have

$$\lim_{r=a_i} \int_0^{s} \frac{\partial g(M,P)}{\partial n_P} ds_M = \lim_{r=a_i} \int_{s-r\varepsilon}^{s} \frac{\partial g(M,P)}{\partial n_P} ds_M$$
$$= \lim_{r=a_i} \int_{\theta_i-\varepsilon}^{\theta_i} \frac{(r_M^2 - a_i^2)\, d\varphi_M}{a_i^2 + r_M - 2\, a_i r_M \cos(\varphi_M - \theta_i)} + \lim_{r=a_i} \eta(r)$$

where $|\eta(r)| \leq r K\varepsilon$, ε being arbitrarily small. From this the third of equations (14) is evident. A similar demonstration applies to the other formulae.

If then we define the function $\Psi(P)$,

$$\begin{aligned} \Psi(P) &= 1, & & P \text{ an interior point of } (0, \theta_i), \\ &= 0, & & P \text{ outside } (0, \theta_i), \\ &= \frac{1}{2}, & & P \text{ at } 0 \text{ or } \theta_i, \end{aligned}$$

we may express $\lim_{r=a_i} I_{0\theta_i}$ as a Daniell integral, in terms of $\Psi(P)$.*

$$\lim_{r=a_i} I_{0\theta_i} = \frac{1}{a_i} \int_{s_i} \frac{1}{\dfrac{\partial g(Q,P)}{\partial n_P}} \Psi(P)\, dU(P),$$

whence

$$U_i(Q_i) = - \int_{(s_i)} \frac{\Psi(P)\, dU(P)}{\dfrac{dh(Q,P)}{d\theta_P}}$$

since $dh(Q,P)/d\theta_P = a_i dh(Q,P)/ds_P = -a_i \partial g(Q,P)/\partial n_P$.

The above integral may be further simplified by remembering that $\Psi(P)$ may be considered as the limit of the sequence of continuous functions which are defined by the equations

$$\begin{aligned} \Psi_n(P) &= 1, & \varepsilon &\leq \varphi_P \leq \theta_i - \varepsilon, \\ &= 0, & \theta_i + \varepsilon &\leq \varphi_P \leq 2\pi - \varepsilon, \end{aligned}$$

and are given linearly in the remaining intervals. In fact, these functions are bounded in their set. Hence, since the

* See Evans, *loc. cit.*, p. 257, or for a systematic study, P. J. Daniell, *A general form of integral*, Annals of Mathematics, vol. 19 (1918), and later papers.

order of integration with respect to $h(Q, P)$ is the reverse of that with respect to θ, we have

$$U_i(\theta_i) = -\int_{h(\theta_i-0)}^{h(0+)} \frac{d\theta_P}{dh(Q, P)} dU(P)$$

$$-\frac{1}{2}\left[\frac{d\theta}{dh(Q, P)}\right]_{\theta_i} \{U(P_{\theta-0}) - U(P_{\theta+0})\}$$

$$-\frac{1}{2}\left[\frac{d\theta_P}{dh(Q, P)}\right]_0 \{U(P_{0-}) - U(P_{0+})\}$$

or, finally,

$$(15) \qquad U_i(\theta_i) = \int_{h(0)}^{h(\theta_i)} \frac{d\theta_P}{dh(Q, P)} dU(P),$$

if $U(P)$ is considered on the Riemann surface corresponding to $h(Q, P)$, where the discontinuities are regular, i. e., where

$$U(P_\theta) - U(P_{\theta-0}) = \frac{1}{2}\{U(P_{\theta+0}) - U(P_{\theta-0})\}, \text{ etc.}$$

By Theorem 6, the unique solution of (15), of limited variation, is given by the equation

$$(16) \qquad U(P_{\theta_i}) - U(P_0) = \int_0^{\theta_i} \frac{dh(Q, P)}{d\theta_P} dU_i(\theta_P),$$

the integrations in (15) and (16) being carried out on the boundary s_i. Consequently we are able to state the following theorem.

THEOREM 7. *The functions $U(P)$, given by* (6) *when the integration is carried out on the Riemann surface for $h(Q, P)$, and the discontinuities of $U(P)$ are consequently regular, and $U_i(\theta_i)$, given by* (10), *satisfy the equations* (15), (16); *so that $U(P)$, except for additive constants, is determined in terms of the $U_i(\theta_i)$, and vice-versa.*

COROLLARY. The equations (15), (16) remain valid if $U(P)$ or the $U_i(\theta_i)$ are absolutely continuous; and if $U(P)$ is absolutely continuous as a function of $h(Q, P)$ the $U_i(\theta_i)$ are all absolutely continuous as functions of their respective arguments, and *vice-versa.*

EXERCISE. Show that a necessary and sufficient condition that $u(M)$ belong to the class of functions considered in Theorems 3, 4 and 5 is that

$$\int_{0\,(g_i=y_i)}^{2\pi} |u(M)|\, dh_i(Q_i, M) < K_i, \qquad i = 0, 1, \cdots, n,$$

in the neighborhood of $y_i = 0$, Q_i being an arbitrary fixed point in the simply connected region of boundary s_i which includes S, and h_i the conjugate to its Green's function.

Show that if

$$W_i(P) = \lim_{y_i=0} \int_{0\,(g_i=y_i)}^{h_i} u(M)\, dh_i(Q_i, M),$$

then

$$U_i(\theta_i) = \int_{h_i(0)}^{h_i(\theta_i)} \frac{d\theta_P}{dh_i(Q_i, P)}\, dW_i(P).$$

$$W_i(P) - W_i(P_0) = \int_0^{\theta_i} \frac{dh_i(Q, P)}{d\theta_P}\, dU_i(\theta_P),$$

the integration being carried out on the boundary s_i.

40. General regions of finite connectivity. Consider a connected open region T in a plane, without isolated point boundaries, and of finite connectivity, say with $n+1$ boundaries. By means of a conformal transformation the region T may be transformed into a particular region S of the kind already considered, bounded by $n+1$ distinct circles, properly situated.* Therefore, whatever results are stated in terms invariant of conformal transformations for the general S with circular boundaries will also hold for T. In particular, Theorem 5 and equation (8) describe such results, interpretable in terms of the accessible points of the boundary of T, as in Chap. V. A special case is that where $U(P)$ is absolutely continuous as a function of $h(Q, P)$ and we may write

* The properties of the conformal transformation are developed in Hurwitz-Courant, *loc. cit.*, p. 322 ff. The region may lie on a Riemann surface provided it is sufficiently like a plane so that any simple closed curve on it divides it into two parts.

$$U(P) - U(P_0) = \int_{h_0}^{h} f(P)\, dh(QP)$$

with $f(P)$ independent of the position of the pole Q. In fact, $f(P)$ represents the boundary values of $u(M)$, which are taken on at almost all accessible points.

The generalized Dirichlet problem is uniquely solvable in the class of functions which are the differences of functions harmonic and not negative in T. In the more specialized problem, the harmonic function of class such that

$$\int_{(g=y)} u(M)\, dh(Q, M)$$

is absolutely continuous, uniformly with respect to y as y approaches 0, is uniquely determined by giving boundary values $f(P)$ almost everywhere on the accessible points of the boundary. The function $f(P)$ is arbitrary provided that it is summable in the Lebesgue sense.

The generalized Neumann problem, which is the important problem of that kind, is uniquely solvable in the class of functions of bounded total flux: i. e., such that

$$\int_{(g=y)} \left| \frac{\partial u}{\partial n} \right| ds < K,$$

as y approaches zero. In this case the function is uniquely determined by the limiting values of

$$\int_{(A)}^{(B)} \frac{\partial u}{\partial n}\, ds$$

as the points A, B, lying on the same curve $g = y$, approach the boundary along curves $h =$ const. The values of the above integral are of course independent of the path in T joining A and B, and its boundary values represent a function of limited variation of one of the end points on the boundary, say P_B. It is sufficient to know this boundary function $F(P_B)$, of limited variation, of total change zero around the boundary, but otherwise arbitrary, merely on the accessible points of continuity for $F(P)$.

By means of the results of Art. 39, we may evidently express the classes of functions, the boundary conditions and the order of accessible boundary points in terms of the $n+1$ associated simply connected regions and their Green's functions. In particular, the kind of resolution given in Theorem 3 may be generalized so as to apply to the general region T. The invariant expressions are given in terms of the Exercise in Art. 39.

A word or two in regard to *isolated* boundary points of regions of finite connectivity is desirable. Here we shall consider merely uniform functions. If a function, single valued and harmonic in the region T, remains bounded in the neighborhood of such a point, say the point 0, Lebesgue's theorem implies that it be harmonic (except for an unnecessary discontinuity) at the point; and therefore for bounded functions the point 0 will not be effectively a boundary point.

If the harmonic function $u(M)$ is not bounded in the neighborhood of 0 it is evident, by means of equation (3), that $u(M)$ may be written as a function which is harmonic in the neighborhood of 0 and at 0, plus a function of the form

$$(17) \quad u_0(M) = k \log r + \sum_1^\infty \frac{1}{r^m} \{a_m \cos m\varphi + b_m \sin m\varphi\},$$

where the part of this expression involved in the summation is the real part of a function which is holomorphic in the entire plane except at 0, and therefore where

$$(18) \qquad \lim_{m=\infty} \sqrt[m]{|a_m|} = \lim_{m=\infty} \sqrt[m]{|b_m|} = 0.\text{*}$$

THEOREM 8. *A necessary and sufficient condition that a single valued function $u(M)$ be harmonic in the neighborhood of 0, except at 0, is that it be the sum of a function harmonic in*

* A direct application of the theorem of Cauchy-Hadamard, that the radius of convergence of the power series $\Sigma \alpha_m \xi^m$ is given by

$$\frac{1}{R} = \lim_{m=\infty} |\alpha_m|^{\frac{1}{m}}.$$

the neighborhood of 0 *and at* 0, *plus a function of form* (17), *where the* a_m, b_m *are subject to* (18).

Consider now functions which are the differences of two functions harmonic and not negative in T, and, in particular, in the neighborhood of the isolated boundary point 0. We shall show first that such a function, if it remains not negative in the neighborhood of 0, can have merely a logarithmic singularity at 0, i. e., will be of the form

$$(19)\qquad u(M) = k\log r + u_1(M),$$

with k a negative constant and $u_1(M)$ harmonic at 0 and in its neighborhood.

In fact, we may, in accordance with Theorem 8, find constants C, k so that the function

$$u'(M) = C + k\log r + \sum_1^\infty \frac{1}{r^p}(a_p\cos p\theta + b_p\sin p\theta),$$

which differs from $u(M)$ by a function harmonic at 0 and in its neighborhood, will be not negative in the neighborhood of 0. Hence for r sufficiently small, we shall have

$$(20)\quad \int_0^{2\pi} u'(M)\,d\theta = \int_0^{2\pi} |u'(M)|\,d\theta = 2\pi C + 2\pi k\log r,$$

so that, in particular,

$$(21)\quad \int_0^{2\pi} u'(M)\,d\theta = \int_0^{2\pi}|u'(M)|\,d\theta < \frac{\varphi_n(r)}{r^n},\quad n = 1, 2, \cdots,$$

with $\lim\limits_{r=0}\varphi_n(r) = 0$.

We have need of the following lemma, which we prove by means of a device described by Poincaré, slightly generalized.*

LEMMA. Consider a function single valued and harmonic in the neighborhood of 0, except at 0, and therefore of the form described in Theorem 8. If

$$(22)\qquad \int_0^{2\pi}|u(r,\theta)|\,d\theta \leq \frac{\varphi(r)}{r^n},$$

* Poincaré, *Théorie du potentiel newtonien*, Paris (1899), p. 208.

where n is a positive integer and $\lim_{r=0} \varphi(r) = 0$, then $a_n = b_n = 0$.

We write

$$X_p = a_p \cos p\theta + b_p \sin p\theta$$

and form, in accordance with (17), the equation

$$\int_0^{2\pi} X_n u(r, \theta) r^n d\theta = \int_0^{2\pi} u_1(r, \theta) X_n r^n d\theta + k \int_0^{2\pi} \log r X_n r^n d\theta$$
$$+ \int_0^{2\pi} \frac{X_n^2 r^n}{r^n} d\theta + \sum{}' \int_0^{2\pi} \frac{X_p}{r^p} X_n r^n d\theta.$$

On account of (22),

$$\left| r^n \int_0^{2\pi} X_n u(r, \theta) d\theta \right| \leqq r^n (|a_n| + |b_n|) \int_0^{2\pi} |u(r, \theta)| d\theta$$
$$\leqq (|a_n| + |b_n|) \varphi(r),$$

and this expression has the limit 0. Moreover

$$\int_0^{2\pi} X_n X_p d\theta = 0, \qquad p \neq n,$$

$$\lim_{r=0} r^n \int_0^{2\pi} X_n u_1(r, \theta) d\theta = 0,$$

$$k r^n \log r \int_0^{2\pi} X_n d\theta = 0,$$

whence

$$\int_0^{2\pi} X_n^2 d\theta = 0,$$

and a_n and b_n must vanish. The lemma is proved.

The function $u'(M)$ satisfies the conditions of this lemma for all positive integral values of n, from (21). Hence

$$u'(M) = C + k \log r,$$

where, of course, $k < 0$. If therefore we return to functions which are the differences of two such not negative functions in the neighborhood of 0, only the same sort of singularity is admitted, and we have the following corollary.*

* Bôcher [Bulletin of the American Mathematical Society, vol. 9 (1903), p. 455] showed that if $u(r, \theta)$ is harmonic $0 < r < R_1$, and becomes $+\infty$ or $-\infty$, for $r = 0$, its only possible singularity at 0 is of the form $k \log r$. See also recent papers by Raynor, ibid. vol. 32 (1926), p. 537, and Kellogg, ibid. vol. 32 (1926), p. 664, and Picone, Lincei (1926).

Corollary 1. If $u(M)$ is the difference of two functions harmonic and not negative in the plane region T, of which 0 is an isolated boundary point, it can have a singularity at 0 merely of the form $k \log r$, and when this term is subtracted the remainder will have merely an unnecessary discontinuity at 0.

By means of the same lemma we have also the following proposition.

Corollary 2. If $\int |u|\, ds$, extended over a circumference of radius r and center 0, remains bounded as r approaches zero, $u(M)$ can have a singularity at 0 merely of the form

$$k \log r + \frac{1}{r}(a_1 \cos\theta + b_1 \sin\theta).$$

This proposition, by further specialization, yields another corollary.

Corollary 3. If $\int |u|\, d\theta$, extended over a circumference of radius r and center 0, remains bounded as r approaches zero, $u(M)$ has merely an unnecessary discontinuity at 0, and 0 is not an effective boundary point.

These corollaries to Theorem 8 show incidentally that the classes of functions, which in S were identical, are no longer co-extensive when one or more of the circles s_i is allowed to shrink down to a point.

41. Annular regions. Determination of the functions $F_0(\theta)$ and $F_1(\theta)$. Let us return now to the region S and assume it to be the annular region between two concentric circles of radii a_0 and a_1, $a_0 > a_1$. We shall take up the problem of determining the functions $F_0(\theta)$, $F_1(\theta)$ of Theorem 3 in terms of the boundary functions $U_0(\theta)$, $U_1(\theta)$.

We write, in accordance with Theorem 3,

$$(23)\qquad \begin{aligned} u(r,\theta) = k \log r + \beta + \frac{1}{2\pi}\int_0^{2\pi} \frac{(a_0^2 - r^2)\, dF_0(\varphi)}{a_0^2 + r^2 - 2a_0 r\cos(\varphi-\theta)} \\ + \frac{1}{2\pi}\int_0^{2\pi} \frac{(r^2 - a_1^2)\, dF_1(\varphi)}{a_1^2 + r^2 - 2a_1 r\cos(\varphi-\theta)} \end{aligned}$$

where, by introducing the constant β, we are able to assume that the $F_0(\varphi)$, as well as the $F_1(\varphi)$, is periodic. We may also assume that these functions vanish for $\varphi = 0$, since they contain an additive arbitrary constant.

THEOREM 9. *In the representation* (23) *the constants* k, β *and the functions* $F_0(\varphi)$, $F_1(\varphi)$, *assumed to be periodic and of limited variation, vanishing at* $\varphi = 0$ *and with regular discontinuities, are uniquely determined by the boundary functions* $U_0(\theta)$, $U_1(\theta)$.

In fact, by taking account of the periodicity of the functions $F_0(\varphi)$, $F_1(\varphi)$ and integrating (23) by parts, we have

$$\int_0^\theta u(r,\theta)\,d\theta = \beta\theta + k\theta\log r$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{a_0^2-r^2}{a_0^2+r^2-2a_0 r\cos(\varphi-\theta)}\right]_{\theta=0}^{\theta=\theta} F_0(\varphi)\,d\varphi$$

$$+\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{r^2-a_1^2}{a_1^2+r^2-2a_1 r\cos(\varphi-\theta)}\right]_0^\theta F_1(\varphi)\,d\varphi,$$

or

$$(24)\quad \begin{aligned} U_0(\theta) &= \beta\theta + k\theta\log a_0 + F_0(\theta) \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{a_0^2-a_1^2}{a_0^2+a_1^2-2a_0 a_1\cos(\varphi-\theta)}\right]_0^\theta F_1(\varphi)\,d\varphi \\ U_1(\theta) &= \beta\theta + k\theta\log a_1 + F_1(\theta) \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi}\left[\frac{a_0^2-a_1^2}{a_0^2+a_1^2-2a_0 a_1\cos(\varphi-\theta)}\right]_0^\theta F_0(\varphi)\,d\varphi. \end{aligned}$$

The equations (24) are mixed integral equations for the determination of the unknowns, the Stieltjes integrals having been eliminated by the integration by parts.

These equations are however of simple type. In fact, if we take $\theta = 2\pi$ we have

$$U_0(2\pi) = 2\pi\beta + 2\pi k\log a_0$$
$$U_1(2\pi) = 2\pi\beta + 2\pi k\log a_1,$$

so that

$$
(25) \qquad \begin{aligned}
k &= \frac{1}{2\pi} \frac{U_0(2\pi) - U_1(2\pi)}{\log(a_0/a_1)}, \\
\beta &= \frac{1}{2\pi} \frac{U_1(2\pi)\log a_0 - U_0(2\pi)\log a_1}{\log(a_0/a_1)},
\end{aligned}
$$

and may therefore be regarded as known in (24).

But now we have merely two simultaneous Fredholm equations for the determination of $F_0(\varphi)$, $F_1(\varphi)$ of the form

$$
(26) \qquad \begin{aligned}
A_0(\theta) &= F_0(\theta) + \int_0^{2\pi} K(\theta, \varphi) F_1(\varphi)\, d\varphi, \\
A_1(\theta) &= F_1(\theta) + \int_0^{2\pi} K(\theta, \varphi) F_0(\varphi)\, d\varphi,
\end{aligned}
$$

in which

$$
(26') \qquad \begin{aligned}
A_0(\theta) &= U_0(\theta) - \beta\theta - k\theta \log a_0, \\
A_1(\theta) &= U_1(\theta) - \beta\theta - k\theta \log a_1,
\end{aligned}
$$

and are periodic, vanish for $\varphi = 0$, are of limited variation and have regular discontinuities.

It is evident therefore that if there are any bounded integrable solutions of (26) they will be periodic and will vanish for $\varphi = 0$. Moreover the total variation of

$$
\int_0^{2\pi} K(\theta, \varphi) F_1(\varphi)\, d\varphi
$$

will be

$$
\leqq N \int_0^{2\pi} d\varphi \int_0^{2\pi} \left| \frac{\partial K(\theta, \varphi)}{\partial \theta} \right| d\theta
$$

if $|F_1(\varphi)| < N$. Hence from the first of equations (26) we see that $F_0(\theta)$ is of limited variation if $F_1(\varphi)$ is bounded. Similarly $F_1(\varphi)$ is of limited variation if $F_0(\varphi)$ is bounded. And if these functions are of limited variation we see from (26) that their discontinuities must be regular. Consequently if (26) has bounded solutions they will be of the type specified in the theorem. The question of the existence and uniquenes

of bounded soulutions depends upon the possible continuous solutions of the corresponding homogeneous equations

$$
(27)\qquad
\begin{aligned}
0 &= G_0(\theta)+\int_0^{2\pi} K(\theta,\varphi)\,G_1(\varphi)\,d\varphi,\\
0 &= G_1(\theta)+\int_0^{2\pi} K(\theta,\varphi)\,G_0(\varphi)\,d\varphi,
\end{aligned}
$$

and the associated homogeneous equations

$$
(28)\qquad
\begin{aligned}
0 &= H_0(\theta)+\int_0^{2\pi} K(\varphi,\theta)\,H_1(\varphi)\,d\varphi,\\
0 &= H_1(\theta)+\int_0^{2\pi} K(\varphi,\theta)\,H_0(\varphi)\,d\varphi.
\end{aligned}
$$

If there are no such solutions of (27) other than zero, there will be none of (28), and *vice versa*; and there will be unique bounded integrable solutions of (26).*

Consider then the equations (28), which may be written explicitly in the form

$$
H_0(\theta) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{a_0^2-a_1^2}{a_0^2+a_1^2-2\,a_0\,a_1\cos(\theta-\psi)}\Bigg]_{\psi=0}^{\psi=\varphi} H_1(\varphi)\,d\varphi,
$$

$$
H_1(\theta) = -\frac{1}{2\pi}\int_0^{2\pi} \frac{a_0^2-a_1^2}{a_0^2+a_1^2-2\,a_0\,a_1\cos(\theta-\psi)}\Bigg]_{\psi=0}^{\psi=\varphi} H_0(\varphi)\,d\varphi.
$$

Accordingly the $H_i(\theta)$ must be periodic with period 2π. Also

$$
\begin{aligned}
H_0(\theta) = &-\frac{1}{2\pi}\int_0^{2\pi} \frac{a_0^2-a_1^2}{a_0^2+a_1^2-2\,a_0\,a_1\cos(\theta-\varphi)} H_1(\varphi)\,d\varphi\\
&+\frac{1}{2\pi}\,\frac{a_0^2-a_1^2}{a_0^2+a_1^2-2\,a_0\,a_1\cos\theta}\int_0^{2\pi} H_1(\varphi)\,d\varphi
\end{aligned}
$$

whence

$$
\int_0^{2\pi} H_0(\theta)\,d\theta = 0.
$$

* The familiar device in the treatment of simultaneous Fredholm equations is to handle them as a single one, by defining a compound kernel over a compound interval [Vivanti, *Elementi della teoria delle equazioni integrali lineari*, Milan (1916), p. 278].

A similar relation holds for $H_1(\theta)$, and we have therefore the equations

$$(29)\quad \begin{aligned} H_0(\theta) &= -\frac{1}{2\pi}\int_0^{2\pi} \frac{a_0^2-a_1^2}{a_0^2+a_1^2-2a_0a_1\cos(\theta-\varphi)} H_1(\varphi)\,d\varphi, \\ H_1(\theta) &= -\frac{1}{2\pi}\int_0^{2\pi} \frac{a_0^2-a_1^2}{a_0^2+a_1^2-2a_0a_1\cos(\theta-\varphi)} H_0(\varphi)\,d\varphi. \end{aligned}$$

Now define the functions

$$\begin{aligned} v_0(r,\theta) &= \frac{1}{2\pi}\int_0^{2\pi} \frac{a_0^2-r^2}{a_0^2+r^2-2a_0 r\cos(\varphi-\theta)} H_0(\varphi)\,d\varphi \\ v_1(r,\theta) &= -\frac{1}{2\pi}\int_0^{2\pi} \frac{r^2-a_1^2}{a_1^2+r^2-2a_1 r\cos(\varphi-\theta)} H_1(\varphi)\,d\varphi, \end{aligned}$$

the first harmonic inside s_0, the second harmonic outside s_1, and regulat at ∞. But if the equations (29) have continuous solutions, we know from Chap. II that both of the harmonic functions, just defined, take on the values $H_0(\theta)$ on s_0 and $-H_1(\theta)$ on s_1, and, being bounded in S, are therefore identical within S. Each is therefore a unique harmonic extension of the other and the function

$$v(r,\theta) = v_0(r,\theta) = v_1(r,\theta)$$

is harmonic in the entire plane and regular at infinity, and therefore must reduce to a constant. But since

$$\int_0^{2\pi} H_0(\varphi)\,d\varphi = 0,$$

this constant must be zero. Hence both $H_0(\theta)$ and $H_1(\theta)$ are identically zero, and the theorem is proved.

The exposition which has just been completed is seen to provide another proof, independent of that in Theorem 5 and its first corollary, that the generalized Dirichlet problem is uniquely solvable for the annular region S in the class of functions of Theorems 3 and 4. For the general region S bounded by $n+1$ circles the consideration of integral equations does not give such simple results. But knowing as we do

that the class of functions described in Theorems 3 and 4 is the same as that of Theorem 5 and equation (8), we know that given the boundary functions $U_i(\theta_i)$ there always exist functions $F_i(\varphi)$ so that the function $u(M)$ will be given by the representation of Theorem 3. In the next section we shall use the integral equations to show that the $F_i(\varphi)$ are essentially uniquely determined.

EXERCISE 1. The reader may show directly that there are no non-zero solutions of (27) by showing that they may be differentiated.

EXERCISE 2. Consider the class of functions $v(r, \theta)$ harmonic within the annular region S such that

$$\int_0^{2\pi} \left| \frac{\partial v}{\partial r} \right| d\theta$$

is bounded for all circles in S concentric with s_0 and s_1. By writing

$$v(r, \theta) = \alpha + k \log r$$
$$- \frac{a_0}{2\pi} \int_0^{2\pi} \log (a_0^2 + r^2 - 2 a_0 r \cos (\varphi - \theta))\, d F_0(\varphi)$$
$$+ \frac{a_1}{2\pi} \int_0^{2\pi} \log (a_1^2 + r^2 - 2 a_1 r \cos (\varphi - \theta))\, d F_1(\varphi)$$

show, by finding the equations to determine $F_0(\varphi)$, $F_1(\varphi)$, that except for an arbitrary additive constant there is a unique solution in the above class of the generalized Neumann problem, for which

$$\lim_{r=a_0} \int_0^{\theta} \frac{\partial v}{\partial r}\, d\theta = V_0(\theta), \quad \lim_{r=a_1} \int_0^{\theta} \frac{\partial v}{\partial r}\, d\theta = V_1(\theta),$$

provided that $V_0(\theta)$, $V_1(\theta)$ are of limited variation, with regular discontinuities, vanish for $\varphi = 0$ and are such that

$$a_0 V_0(2\pi) = a_1 V_1(2\pi).$$

The integral equations for $F_0(\varphi)$, $F_1(\varphi)$ reduce to those already treated.

42. Uniqueness of the representation of Theorem 3 for S. We shall prove the following Theorem.

THEOREM 10. *Given $u(M)$ as in Theorems 3 and 4, we may write*

$$(30)\quad \left\{\begin{aligned} u(r,\theta) &= \beta + \sum_1^n k_i \log r_i \\ &\quad + \frac{1}{2\pi}\int_0^{2\pi} \frac{(a_0^2 - r_0^2)\, d F_0(\varphi)}{a_0^2 + r_0^2 - 2 a_0 r_0 \cos(\varphi - \theta_0)} \\ &\quad + \sum_1^n \frac{1}{2\pi}\int_0^{2\pi} \frac{(r_i^2 - a_i^2)\, d F_i(\varphi)}{a_i^2 + r_i^2 - 2 a_i r_i \cos(\varphi - \theta_i)} \end{aligned}\right.$$

with the functions $F_0(\varphi)$, $F_i(\varphi)$ as in Theorem 3, but vanishing at $\varphi = 0$, with regular discontinuities, and all of them periodic with period 2π. With these conditions the functions $F_0(\varphi)$, $F_1(\varphi), \cdots, F_n(\varphi)$ and the constants $\beta, k_1, \cdots, k_n$ are all uniquely determined.

Corresponding to equations (24) of Art. 41 we have the equations

$$(31)\quad \left\{\begin{aligned} & U_0(\theta_0) = \beta\,\theta_0 + \sum_1^n k_i \int_0^{\theta_0} \log r_{i0}\, d\theta_0 + F(\theta_0) \\ & + \frac{1}{2\pi}\sum_1^n \int_0^{2\pi} \left\{ \int_0^{\theta_0} \frac{(r_{i0}^2 - a_i^2)\, d\theta_{i0}}{a_i^2 + r_{i0}^2 - 2 a_i r_{i0} \cos(\varphi - \theta_{i0})} \right\} d F_i(\varphi) \\ & U_j(\theta_j) = \beta\,\theta_j + k_j \theta_j \log r_j + \sum_1^n{}' k_i \int_0^{\theta_j} \log r_{ij}\, d\theta_j + F_j(\theta_j) \\ & + \frac{1}{2\varphi}\int_0^{2\pi} \left\{ \int_0^{\theta_j} \frac{(a_0^2 - r_{0j}^2)\, d\theta_{0j}}{a_0^2 + r_{0j}^2 - 2 a_0 r_{0j} \cos(\varphi - \theta_{0j})} \right\} d F_0(\varphi) \\ & + \frac{1}{2\pi}\sum{}' \int_0^{2\pi} \left\{ \int_0^{\theta_j} \frac{(r_{ij}^2 - a_i^2)\, d\theta_{ij}}{a_i^2 + r_{ij}^2 - 2 a_i r_{ij} \cos(\varphi - \theta_{ij})} \right\} d F_i(\varphi). \end{aligned}\right.$$

$$j = 1, 2, \cdots, n,$$

in which θ_{ij}, r_{ij} are the polar coördinates of a point on s_j with respect to the center of the circle s_i.

Since the equations (30) are necessary and sufficient that $u(M)$ be of the given class, and the member of the class is uniquely determined by the boundary functions $U_i(\theta_i)$, $i = 0, 1, \cdots, n$, the equations (31) are necessary and

sufficient for (30), where the $U_i(\theta_i)$ are of limited variation, with regular discontinuities and vanish when their arguments vanish. Suppose there where two sets of solutions of the kind specified in Theorem 10; the set of differences β, k_i, F_0, F_i between the corresponding elements of the two sets of solutions would be solutions of the homogeneous Stieltjes integrals of the same kind.

From the form of these equations, since the boundary circles are all distinct, we have a right to differentiate under the integral signs, and we see that the functions $F_0, F_1, \cdots, F_n$ are all absolutely continuous. Hence we may differentiate the equations and write

$$
(32)\quad \begin{cases}
0 = \beta + \sum_1^n k_i \log r_{i0} + f_0(\theta_0) \\
\qquad + \dfrac{1}{2\pi}\sum_1^n \displaystyle\int_0^{2\pi} \dfrac{(r_{i0}^2 - a_i^2)\, f_i(\varphi)\, d\varphi}{a_i^2 + r_{i0}^2 - 2\, a_i r_{i0} \cos(\varphi - \theta_{i0})} \\
0 = \beta + \sum_1^n k_i \log r_{ij} + f_j(\theta_j) \\
\qquad + \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \dfrac{(a_0^2 - r_{0j}^2)\, f_0(\varphi)\, d\varphi}{a_0^2 + r_{0j}^2 - 2\, a_0 r_{0j} \cos(\varphi - \theta_{0j})} \\
\qquad + \dfrac{1}{2\pi}\sum_1^{n}{}' \displaystyle\int_0^{2\pi} \dfrac{(r_{ij}^2 - a_i^2)\, f_i(\varphi)\, d\varphi}{a_i^2 + r_{ij}^2 - 2\, a_i r_{ij} \cos(\varphi - \theta_{ij})}, \\
\qquad\qquad j = 1, 2, \cdots, n,
\end{cases}
$$

where

$$
F_0(\varphi) = \int_0^{\varphi} f_0(\varphi)\, d\varphi, \qquad F_j(\varphi) = \int_0^{\varphi} f_j(\varphi)\, d\varphi,
$$

and the functions $f_0(\varphi)$, $f_j(\varphi)$ are evidently continuous.

Define now the functions

$$
v_0(M) = \beta + \frac{1}{2\pi}\int_0^{2\pi} \frac{(a_0^2 - r_0^2)\, f_0(\varphi)\, d\varphi}{a_0^2 + r_0^2 - 2\, a_0 r_0 \cos(\varphi - \theta_0)}
$$

$$
v_j(M) = k_j \log r_j + \frac{1}{2\pi}\int_0^{2\pi} \frac{(r_j^2 - a_j^2)\, f_j(\varphi)\, d\varphi}{a_j^2 + r_j^2 - 2\, a_j r_j \cos(\varphi - \theta_j)}, \qquad j = 1, 2, \cdots, n.
$$

The function

$$v(M) = v_0(M) + \sum_1^n v_j(M)$$

is harmonic in S, and bounded, and by (32) vanishes everywhere on the boundary of S; hence $v(M) = 0$ identically in S.

We have

$$v_0(M) = -\sum_1^n v_j(M) \tag{33}$$

in S, and accordingly the left hand member provides the unique extension of the right hand member everywhere inside s_0, and the right hand member provides the unique extension of the left hand member outside $s_1, \cdots, s_n$. We thus define the function $V_0(M)$ in terms of (32) and these extensions. Hence

$$\int_0^{2\pi} V_0(M)\, d\theta = 2\pi V_0(0) = 2\pi\beta.$$

But this integral may also be obtained by integrating over a large circle of radius R, and yields therefore

$$2\pi\beta = \int_0^{2\pi} \sum_1^n k_i \log R\, d\theta_0 + \int_0^{2\pi} \sum_1^n k_i \log \frac{r_i}{R}\, d\theta_0.$$

The second integral comes as near to the value 0 as we please, by taking R large enough, while the first is $(2\pi \sum k_i) \log R$. Hence we must have $\sum k_i = 0$, $\beta = 0$.

Now however the function $V_0(M)$ is harmonic all over the plane and regular at ∞. It is therefore a constant, and since its integral from 0 to 2π is zero, it must vanish identically; i. e.,

$$v_0(M) \equiv 0, \qquad f_0(\varphi) \equiv 0.$$

An analogous method of procedure shows in turn that k_1, $f_1(\varphi)$, k_2, $f_2(\varphi)$, $\cdots$, k_n, $f_n(\varphi)$ all vanish. Hence the $F_0(\varphi)$, $F_i(\varphi)$ also vanish identically, and the theorem is proved.

CHAPTER VII

RELATED PROBLEMS

43. A simple discontinuous boundary value problem. Consider a region T bounded by a simple rectifiable curve s, with a unique tangent at each point; and consider a function $f(s)$, defined on the boundary in terms of the arc length, continuous except at a finite number of points $A_1, \cdots, A_n$, and at each of these points such that both $f(s_i+0) = b_i$ and $f(s_i-0) = a_i$ exist. It is easy to show that there is one and only one function $u(M)$ bounded and harmonic in S which takes on the values $f(s)$ continuously as M approaches any point of the boundary not an A_i.*

In fact, the function

$$\alpha(M) = -\sum_1^n \frac{b_i - a_i}{\pi} \arctan \frac{y - y_i}{x - x_i}$$

is bounded and harmonic in S and takes on, on the boundary, values $\varphi(s)$ which have precisely the same discontinuities as $f(s)$. If then we denote by $\beta(M)$ the solution of the continuous boundary value problem $f(s) - \varphi(s)$, the function

$$u(M) = \alpha(M) + \beta(M)$$

will be a solution of our problem.

That the solution is unique follows of course from the theory of Chap. V. But the uniqueness is established also by elementary means on the basis of a theorem of Zaremba.† If $U(M)$, harmonic in T, takes on continuously the boundary values 0 except at a finite number of points A_i, where

* Goursat, *Cours d'analyse,* vol. III, Paris (1915), p. 204.

† Bulletin of the Academy of Sciences of Cracow (1909), p. 561. See Goursat loc. cit. The theorem may be regarded as containing as a special case the introductory theorem of Chapter IV, there ascribed to Lebesgue, as a special case again of other results of his.

$$\lim_{r_i=0}\left|\frac{U(M)}{\log r_i}\right| = 0, \qquad r_i = \overline{A_i M},$$

then $U(M)$ vanishes identically in T. The proof of this proposition may be omitted since a slightly more general theorem will be discussed, in Art. 47, which depends on the same method of proof.

In the simple problem, just discussed, it was feasible to lead the question back to one where the boundary values were continuous. This remark, in fact, applies to the usual treatment of a discontinuous boundary value problem, which thus becomes a limiting case, as in the theorems on sequences cited in Chap. V, or a mere offshoot, as in the example just given. This point of view is hardly due to the physical applications of the problem, for it cannot be said that modern physics has any special affinity for the continuous, if indeed a science can be called modern which is entirely Greek in its devotion to whole numbers.* It is therefore a special strength of the Stieltjes integral that it applies the fundamental processes of integration in the representation of discontinuous quantity.

In the treatment given in these chapters the emphasis has been put directly on functions of limited variation, rather than continuous functions, on account of the relation of the former to general distributions of matter. But this statement implies more than can rightfully be asserted, since except for the simplest kinds of boundaries it was found desirable to introduce the Green's function and the theory of conformal representation. The latter is a theory which requires modification for more than two dimensions. And the Green's function brings in to some extent the continuous boundary value problem—the determination of a harmonic function which takes on the frontier values *log r*. On the other hand, with this much admitted, there is something gained in reducing the problem to central ideas, where so much is known in detail.

* See the first two pages of Born, *Problems of atomic dynamics*, Cambridge (1926).

44. Continuous boundary value problems. Given a function $f(M)$ continuous in an open region T and on its frontier, the central problem is to find a function $u(M)$, harmonic in T, which takes on the value $f(P)$ continuously, as M, in T, approaches the point P, on the boundary. It is only necessary to mention the alternating method of Schwarz, Poincaré's "méthode de balayage", the minimal processes of Riemann, Hilbert, Zaremba and Lebesgue, the successive approximation method of Le Roux, and Phillips and Wiener, reducing the approximate solution of the problem to the solution of a set of linear algebraic equations, and the method of C. Neumann which culminates in Fredholm's solution of the linear integral equation with constant limits.* These provide solutions of the problem, in part, or as a whole.

The method of Poincaré, for example, applies in particular to any region bounded by a finite number of distinct simple closed curves, with continuously turning tangent at every point. Now if the region T is a bounded open plane region whose frontier consists of a finite number of distinct connected sets, these boundary sets, being closed, will be at a finite distance from each other. Hence T may be regarded as the limit of regions T_n of the sort to which the Poincaré method applies, all of the same connectivity as T, and such that T_{n+1} includes T_n with its boundary. The Green's function $g_n(Q, M)$ of T_n is for each M an increasing function of n which does not exceed $G(Q, M)$, where $G(Q, M)$ is the Green's function for a circle of center Q and radius sufficiently large; hence g_n approaches a limit function $g(Q, M)$ which is evidently independent of the mode of formation of the T_n. This function, defined as 0 for all points on the boundary of T, may be called the Green's function for T.

* Summaries of these matters are given in G. Bouligand, *Fonctions harmoniques. Principes de Picard et de Dirichlet* [Mémorial des sciences mathématiques, fasc. XI] Paris (1926), and in the forthcoming report of O. D. Kellogg on the continuous boundary value problem, delivered as a Symposium before the American Mathematical Society, Dec. 1925. See also Goursat, *Cours d'Analyse*, vol. III.

By means of Harnack's theorem we know that $g(Q, M)$ is the sum of $\log 1/r$ and a function harmonic at all points of T. That it takes on continuously the boundary values 0 at all boundary points P is a theorem of Osgood.* The theorem is established by means of an elementary conformal transformation which carries T into a portion of the interior of a circle and, at the same time, the given boundary point P into a point of the circumference. In this form the demonstration applies directly to a simply connected region, but the theorem is seen to be valid for our region T since the Green's function for T with pole Q cannot exceed the Green's function with pole Q for any simply connected region containing T which has a part of the boundary of T for its boundary. That the method of proof is not accidental is evident from the fact that the theorem is not necessarily true for three-dimensional regions, or for plane regions of infinite connectivity.

Some of the methods devised for the continuous boundary value problem apply also to some extent to discontinuous boundary values. The method of integral equations may be so used for simple types of boundaries. More suggestive is the method of Zaremba.† He considers the quantity

$$(1) \qquad B(f, h) = \int_T \left(\frac{\partial f}{\partial x}\,\frac{\partial h}{\partial x} + \frac{\partial f}{\partial y}\,\frac{\partial h}{\partial y}\right) d\sigma.$$

in which $f(M)$ is the given continuous function and $h(M)$ is an arbitrary function, harmonic in T, subject merely to the conditions that $A(f)$ and $A(h)$ both exist, where

$$(2) \qquad A(f) = \int_T \left\{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right\} d\sigma.$$

* Osgood, *Funktionentheorie*, Leipzig (1912), p. 702.

† Fourth International Mathematical Congress, vol. 2, p. 194, Bulletin of the Academy of Sciences of Cracow (1909), p. 197. We have at hand merely the summaries given in the Jahrbuch der Fortschritte der Mathematik (1909) and in Lichtenstein's article in the Encyklopädie.

Take now functions $v(M)$, not necessarily harmonic in T, but such that $A(v)$ is defined, which satisfy a boundary condition which can be expressed in the form

$$B(v, h) = B(f, h)$$

for all functions $h(M)$ of the class specified. That this is a sort of average value boundary condition is seen by applying Green's theorem. The solution of the problem depends upon finding, by means of sequences of functions $v(M)$, a particular function $v(M)$ which gives to the integral $A(v)$ its minimum value.

This method of solution does not yield a result in some cases even with simple boundaries, such as circles. Thus with values given continuously on the circumference of a circle, a corresponding function $f(M)$ can be found, continuous within the circle and on the circumference, for such a function is given by Poisson's integral. But it may not be possible to give such a function which will satisfy the condition that $A(f)$ exists.* On the other hand the method does yield a result in many cases where the boundary values are discontinuous, and is thus a direct attack on the discontinuous boundary value problem.

Some of the methods cited are special cases of a general treatment of the solution of the Dirichlet problem as a functional of the boundary curve of the region and of the values on that curve. For boundaries which are not simple the solution appears as a continuous prolongation of the functional.†

45. **Regions with continuous boundaries.** A special case of the continuous boundary value problem is that of the conformal transformation of a circle into a region bounded by a simple continuous curve. It is a problem which is all the more special on account of the fact that two conjugate harmonic

* Hadamard, Bulletin de la Societé Mathématique de France, vol. 34 (1906).

† The most systematic discussion of this point of view will be found in three articles by G. Bouligand, Bulletin des Sciences mathématiques, vol. 48 (1924), pages 183, 205, 246.

functions are at the same time taking on continuous boundary values. We have already had an instance of such specialization (Ex. 5, Art. 27). More remarkable is the theorem of Fejér: *if $w(z)$ maps a circle into the interior of a simply covered plane region bounded by a Jordan curve, the Taylor's series for $w(z)$, with respect to the center of the circle, converges uniformly, not only in the interior, but also on the circumference of the circle.** In other words, the conjugate Fourier series for the boundary values $f(\theta)$, $g(\theta)$ are uniformly convergent, $0 \leqq \theta \leqq 2\pi$.

The theorem rests upon the following lemma. Let

$$\sum_{0}^{\infty} u_i = u_0 + u_1 + \cdots + u_n + \cdots, \tag{3}$$

the terms being real or complex, be summable by the method of the arithmetic mean; if, further, the series

$$\sum_{1}^{\infty} i\,|u_i|^2 = |u_1|^2 + 2\,|u_2|^2 + \cdots + n\,|u_n|^2 + \cdots, \tag{4}$$

is convergent, the series (3) is also convergent. If (3) is summable uniformly by the method of the arithmetic mean, and (4) is convergent, the series (3) is uniformly convergent.

We know also, as a classical theorem, that if $f(\theta)$ is a summable function in the Lebesgue sense its Fourier series is summable by the method of the arithmetic mean for any θ where $f(\theta)$ is the derivative of its indefinite integral, and uniformly through any closed interval where $f(\theta)$ is continuous.

Now we have noticed that the integral

$$I = \int\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right\} d\sigma = \int\left\{\left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \theta}\right)^2\right\} d\sigma,$$

if it exists, extended over the interior of the circle, is the area of the region into which the circle is transformed by

* Comptes Rendus, t. 156 (1913), p. 46, also in *Mathematische Abhandlungen H. A. Schwarz zu seinem fünfzigjährigen Doktorjubiläum gewidmet von Freunden und Schülern,* Berlin (1914), p. 42.

means of the conformal transformation defined by the harmonic function u and its conjugate. If we write

$$u(r, \theta) = \frac{1}{2} a_0 + \sum_1^\infty r^m (a_m \cos m\theta + b_m \sin m\theta)$$

this integral for the unit circle has the value

$$I = \lim_{r=1} \pi \sum_1^\infty m r^{2m} (a_m^2 + b_m^2);$$

and if I remains bounded in the limit, the series of positive terms

$$\pi \sum_1^\infty m (a_m^2 + b_m^2) \tag{5}$$

is convergent; and conversely, if the series is convergent, the limit I exists, and represents this area. Moreover the convergence of (5) is also sufficient that the Fourier series

$$\frac{1}{2} a_0 + \sum_1^\infty (a_m \cos m\theta + b_m \sin m\theta)$$

shall be summable by the method of the arithmetic mean, almost everywhere, to a function $f(\theta)$ for which the a_m, b_m are the Fourier coefficients. In fact the existence of

$$\sum_1^\infty (a_m^2 + b_m^2)$$

is sufficient that $f(\theta)$ be summable in the Lebesgue sense, with its square.

If therefore the area of the region into which the circle is transformed, measured on the leaves of the Riemann surface, if necessary, remains finite, the lemma tells us that the two conjugate Fourier series converge almost everywhere on the circumference to functions $f(\theta)$, $g(\theta)$ such that

$$\lim_{r=1} u(r, \theta) = f(\theta), \qquad \lim_{r=1} v(r, \theta) = g(\theta). \tag{6}$$

A special case of this situation is that where $u(r, \theta)$, $v(r, \theta)$ are bounded and the transformed region is a simply covered plane region, or is spread on a Riemann surface of a finite number of leaves.

Suppose finally that the transformation maps the circle on a plane region interior to a simple closed continuous curve. Carathéodory's theorems, already cited, tell us that the transformation is continuous within and on the boundary, and therefore that the boundary values of the u and v are two continuous functions $f(\theta)$, $g(\theta)$. Hence the summation by the method of the arithmetic mean converges uniformly, and the theorem is proved.

Exercise. Interpret Fejér's theorem with respect to monotonic changes of variable in a given arbitrary continuous function $f(x)$, and development in Fourier series.

46. Regions with rectifiable boundaries. If the boundary values of the conjugate functions u, v, harmonic within the unit circle, are functions of limited variation of some parameter t on the boundary of the transformed region, they will also be functions of limited variation of θ; for they will be functions of limited variation of the arc-length s, and the latter is a monotonic function of θ. Hence the conditions given in Example 5, Art. 27 that

$$\int_0^{2\pi}\left|\frac{\partial u(r,\theta)}{\partial r}\right| d\theta, \qquad \int_0^{2\pi}\left|\frac{\partial u(r,\theta)}{\partial \theta}\right| d\theta$$

remain bounded as r approaches 1, or that $u(r,\theta)$, $v(r,\theta)$ be functions of θ of limited variation, uniformly in r, $r<1$, are necessary and sufficient that the boundary of the transformed region be a rectifiable curve.

We wish to prove that *the transformation which carries the interior of the unit circle conformally into the interior of a region with rectifiable boundary, simply covered on a plane or on a Riemann surface, not necessarily of a finite number of leaves, is conformal almost everywhere on the boundary.* Incidentally the boundary values f, g will be seen to be *absolutely continuous* functions of θ, as they must necessarily be of s.

Consider a point P' on the circumference where $f'(\theta)$ and $g'(\theta)$ both exist, and $f'(\theta) \neq 0$. For the corresponding point P

of the boundary of the transformed region T the slope of the tangent is defined and is equal to

$$\frac{dv}{du} = \frac{g'(\theta)}{f'(\theta)} = \tan \beta.$$

Let s_1' be an arc in S which leads to P'; the corresponding arc s_1 in T leads to P (which may of course be a part of a multiple point of the boundary). We suppose that the direction of the arc s_1' is defined at every point, and changes continuously as its generic point M' approaches and coincides with P', making at P' an angle α', which is not zero,

$$\tan \alpha' = -\frac{dr}{r\,d\theta},$$

with the tangent direction.

On the curve s_1 we have

$$\frac{dv_1}{du_1} = \frac{\dfrac{\partial v}{\partial r}\,dr + \dfrac{\partial v}{\partial \theta}\,d\theta}{\dfrac{\partial u}{\partial r}\,dr + \dfrac{\partial u}{\partial \theta}\,d\theta}$$

But since $\partial u/\partial r$, $\partial u/\partial \theta$, $\partial v/\partial r$, $\partial v/\partial \theta$ approach their values at P' when M' approaches P' in the wide sense, by Art. 23, Chap. IV, we have

$$\lim_{M'=P'} \frac{dv_1}{du_1} = \frac{\dfrac{g'(\theta)}{f'(\theta)} - \dfrac{dr}{r\,d\theta}}{1 + \dfrac{g'(\theta)}{f'(\theta)}\,\dfrac{dr}{r\,d\theta}} = \tan(\beta + \alpha'),$$

when $dr/d\theta$ approaches a finite value, and

$$\lim_{M'=P'} \frac{dv_1}{du_1} = -\frac{f'(\theta)}{g'(\theta)}$$

when $dr/d\theta$ approaches $\mp\infty$. Hence in all cases

$$\alpha' = \arctan\left(\frac{dv_1}{du_1}\right)_P - \beta$$

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and the transformation is seen to be conformal at P'. Similar reasoning is valid if $g'(\theta)$ is not zero at P'.

Exercise 1. Let M'_i be a sequence of points in S and M_i the corresponding sequence in T. Show that a necessary and sufficient condition that M_i approach P and the direction of $M_i P$ approach a limiting direction not tangent to the boundary of T is the existence of the same situation in S, provided that $f'(\theta)$ or $g'(\theta)$ does not vanish at P'.

But the points where $f'(\theta)$ or $g'(\theta)$ fails to exist form a set of measure 0 on the circumference. The points where $f'(\theta)$ and $g'(\theta)$ both exist and vanish are points at which the transforming function $w(z)$ has a derivative $w'(z)$ which takes on the value 0 in the wide sense. Here we can make use of a remarkable theorem of Lusin and Priwaloff.* *The function $w'(z)$, holomorphic within the circle, cannot vanish on the circumference, in the wide sense, at a set of more than zero measure.* Hence the transformation defined by $w(z)$ is conformal for almost all θ.

The result may be made still more precise. In the above cited memoir the authors show that in the conformal transformation of the interior of the circle into a simply covered region with rectifiable boundary s, any point set of zero measure on the circumference will correspond to a set of zero measure on s.† The same is true for the general simply connected region with rectifiable boundary. In fact, if there were a set of positive measure on s, corresponding to a set of zero measure on the circumference, there would be such a set in one of the leaves of the Riemann surface, since the number of leaves is denumerable; and there would be a portion of this set, of positive measure, contained between two h-curves of finite length, drawn from a point O.

But on the circle, the corresponding set would be of zero

* Annales scientifiques de l'École Normale Supérieure, t. 41 (1925), p. 159.

† Lusin and Priwaloff, loc. cit., p. 156. Reference is there made also to earlier papers: Lusin, *Sur la représentation conforme*, Bull. de l'Inst. Pol. Ivanovo-Vosn. (1919), and Priwaloff, *Intégrale de Cauchy*, Bull. de l'Univers. á Saratov. (1918).

measure, lying on an arc contained between two circles from a point $0'$, drawn normally to the circumference. If we map this portion of the circle on the whole circle, the set on the arc will become a set on the circumference, still of zero measure; and by means of a transformation which is the succession of the inverses of the two just indicated, taken in the proper order, we shall transform the whole circle into the simply covered region whose rectifiable boundary is made up of a portion of s and the h-curves from 0. Thus we arrive at a contradiction with the theorem cited.

The coördinates of a point on the boundary are thus seen to be absolutely continuous functions of θ as well as s. If we combine this result with that of Example 5, Art. 27, we notice that *if two conjugate Fourier series represent functions of limited variation, those functions must be absolutely continuous.* Finally, we are able to say that the transformation is conformal on the boundary except possibly at a set of zero measure, measured on s. This was the theorem to be proved.

At any boundary point P for T where the corresponding values of $f'(\theta)$ and $g'(\theta)$ are defined and not both zero, we may speak about approach to the boundary *in the wide sense.* For if we consider the transformation into the circle, any curve which leads to P, eventually coming and remaining within an angle at P whose sides make positive angles with the tangent, corresponds to a curve in the circle, leading to P', which eventually comes and remains within any angle whose sides make smaller angles with the tangent at P'—and *vice versa.* Hence we can generalize at once Fatou's theorem to regions with rectifiable boundaries, considering approach in the wide sense.

Let $f(s)$ be any function summable in the Lebesgue sense on s, and $F(s)$ a function of limited variation on s; they will then have the same properties with respect to θ, since s is a monotonic absolutely continuous function of θ. Hence the Dirichlet and Neumann problems may be stated directly in terms of $f(s)$ and $F(s)$. In particular, and this is the generalization of Fatou's theorem, *if* $\zeta(M)$ *is harmonic in* T

and of class (i) *it takes on boundary values almost everywhere on s in the wide sense; in the subclass* (ii) *it is determined by those boundary values.**

For the Neumann problem we must have $F(s)$ periodic or

$$\int_s f(s)\,ds = 0.$$

Let $\zeta(M)$ be of the class (j). Then $\partial\zeta/\partial g$, taken along curves $h =$ const., takes on its boundary values in the wide sense, almost everywhere. But if we denote by n the direction of the interior normal to the curves $g =$ const., we have

$$\frac{\partial\zeta}{\partial n} = \frac{\partial\zeta}{\partial g}\frac{\partial g}{\partial n} = \frac{\partial\zeta}{\partial g}\bigg/\left(r^2\left(\frac{\partial u}{\partial r}\right)^2 + \left(\frac{\partial u}{\partial\theta}\right)^2\right)^{\frac{1}{2}},$$

where $w = u + iv$ maps the circle on T. Hence $\partial\zeta/\partial n$ *takes on its boundary values almost everywhere in the wide sense on s; and if $\zeta(M)$ is of the subclass* (jj) *it is determined by those boundary values.* It is in this sense that the Neumann problem is particularly related to regions with rectifiable boundaries, as distinguished from the generalized problem treated in Chap. V.

From the considerations of Chap. VI, we see that the theorems on these classes of functions apply to plane regions of finite connectivity bounded by $n+1$ rectifiable distinct curves. The conformal transformations involved refer to the simply connected regions determined by the boundaries separately.

Exercise 2. Show that θ is also an absolutely continuous function of s.

47. Regions of infinite connectivity. Consider a plane region T, bounded externally by a simple closed rectifiable curve C_0, and internally by a denumerable infinity of curves of the same kind, $C_1, C_2, \cdots$. The complete boundary of T consists of the set of points formed by these curves and all the limiting points of them. If $u(M)$, bounded and harmonic

* Plancherel, Bull. des Sc. Math., vol. 34 (1910), p. 111, for more special boundaries shows that if $\lim_{g\ 0}\int_0^h |u|\,ds \quad \int_0^h |f(s)|\,ds$, there can be not more than one solution.

in T, takes on the boundary values 0 continuously on C_0, $C_1, C_2, \cdots$ does it vanish identically in T?

Let E_k be the collection of limiting points of the set consisting of the points of

$$C_k + C_{k+1} + \cdots.$$

It is a closed set which includes all the points of C_k, $C_{k+1}, \cdots$. Define also

$$E = \lim_{k=\infty} E_k = E_0 \cdot E_1 \cdot E_2 \cdots,$$

which is also a closed set. The complete boundary of T is the set

$$E + C_1 + C_2 + \cdots.$$

A partial answer to our problem may now be stated as follows: *With the given hypotheses, if E is a reducible set, $u(M)$ vanishes identically in T.* We assume that E is finitely reducible.

In order to prove the proposition, let Q be an arbitrary point of the open region T. We shall show that if $\eta > 0$ is given, $|u(Q)| < \eta$. Suppose that E has successive derived sets E', E'', $\cdots$, $E^{(l)}$, $E^{(l)}$ consisting merely of a finite number of points

$$A_1^{(l)}, A_2^{(l)}, \cdots, A_{n_l}^{(l)}.$$

With these points as centers we draw circles of radii $\varrho_i^{(l)}$, respectively, small enough so that Q is exterior to them, and so that no point of $E^{(l-1)}$ lies on any circumference; but otherwise arbitrary. Outside of these circles there will remain a finite number of points of $E^{(l-1)}$, say

$$A_1^{(l-1)}, A_2^{(l-1)}, \cdots, A_{n_{l-1}}^{(l-1)}.$$

With these points as centers we draw circles of radii $\varrho_i^{(l-1)}$, respectively, distinct from the previous ones, small enough so that Q is exterior to them, and so drawn that no point of $E^{(l-2)}$ lies on any circumference; but otherwise arbitrary. Proceeding in this way we finally isolate all the points of E.

Let T' be the connected portion of T, exterior to these circles and containing Q. There will be only a finite number

of the C_k, say $C_1, \cdots, C_q$, which have points outside or on the circumferences of these circles. The region which is defined by the curves $C_0, C_1, \cdots, C_q$, as complete boundary, is finitely connected, and the constructed circles either insert additional distinct boundaries or form cross or return cuts in this region. Hence the connectivity of T' is finite, and and it has no isolated point boundaries.

Let H be the diameter of T, and write

$$\eta' = \eta'' \cdots = \eta^{(l)} < \eta/(l+1)$$

Choose now the sets of constants, defined successively for $j = l, l-1, l-2, \cdots, 1,0$, as follows:

$$(7) \quad \begin{cases} \eta_i^{(j)} = \eta^{(j)}/n_j, \quad \varepsilon_i^{(j)} = \eta_i^{(j)}/\log \dfrac{H}{\overline{A_i^{(j)} Q}} \\ \varrho_i^{(j)} \text{ small enough so that } |u(M)| < \varepsilon_i^{(j)} \log H/\varrho_i^{(j)} \\ \text{when } M \text{ is on the circle of center } A_i^{(j)} \text{ and radius } \varrho_i^{(j)}. \end{cases}$$

The function

$$(8) \qquad v(M) = \sum_{j=1}^{l} \sum_{i=1}^{n_j} \varepsilon_i^{(j)} \log H/\overline{A_i^{(j)} M}$$

is harmonic in T' and continuous in the region consisting of T' and its boundary points. It is moreover everywhere positive in T'. But on the boundary of T', $|u(M)| < v(M)$. Hence throughout T',

$$-v(M) < u(M) < v(M),$$

so that

$$|u(Q)| < v(Q) = \sum_{j=1}^{l} \sum_{i=1}^{n_j} \eta_i^{(j)} < \eta. \qquad \text{[Q. E. D.]}$$

For what has so far been obtained the hypothesis that the boundaries $C_0, C_1, \cdots$ be rectifiable is unnecessary; in fact, beyond the requirement that any finite number of them counting from C_0, $C_0 + C_1 + \cdots + C_q$, form the boundary of a bounded finitely connected region without isolated point boundaries, it is merely assumed that E be reducible.

A different kind of theorem is however not so easily extended to general regions.

Let us assume the C_i rectifiable again, but instead of requiring the $u(M)$ to take on the value 0 continuously, let us suppose merely that the function, remaining bounded, takes on the value 0 on each C_i almost everywhere in the narrow sense, that is, as M approaches the boundary along the curves $h_i(Q, M) = \text{const.}$, where h_i is the conjugate of the Green's function for the region bounded by C_i and containing T. Then *if E is reducible, $u(M)$ vanishes identically in T.* This is a discontinuous boundary value problem.

From Art. 40 we know that we can express $u(M)$ in T' by means of the Green's function integral, which now takes this form (see Art. 39)

$$u(M) = \frac{1}{2\pi}\int_s u(P)\frac{\partial g(M, P)}{\partial n_P}\, ds_P$$

in terms of its boundary values. The same applies to the function $v(M)$ given by (8). By comparing these explicit expressions we find again $|u(Q)| < v(Q) < \eta$, and thus prove the theorem.

Exercise. Replace the condition that $u(M)$ be bounded by a condition of uniform absolute continuity with respect to each C_i, assuming that $u(M)/\log PM$ has the limit 0 as M approaches P, P being a point of E.

We have been concerned here with a set E of *zero capacity.** Capacity is measured by

$$\frac{1}{2\pi}\int_s \frac{\partial u}{\partial n}\, ds.$$

where s is any curve or system of curves enclosing E, and u is the lower bound of all conductor potentials which take the value 1 on curves s enclosing E. Incidentally, a function harmonic in a certain region except at a set of interior points of zero capacity has only unnecessary discontinuities, if it

* Wiener, Journal of Mathematics and Physics of the Massachusetts Institute of Technology, vol. 3 (1924), p. 26.

remains bounded. In fact, if we denote by B a closed set of frontier points of T which are such that $B+T$ is still an open region, a necessary and sufficient condition that

$$\lim_{M=P} G(M, P) > 0$$

is that B be of capacity 0.* The same concept is valuable for the discussion of the regularity or irregularity, as far as the continuous boundary value problem is concerned, of arbitrary boundary points.†

Let us in conclusion sketch the proof of a general theorem. Let $f_i(P)$ be assigned, bounded, in numerical value less than a fixed number N, and integrable in the Lebesgue sense, on C_i, $i = 0, 1, \cdots$. Let $u_0(M)$ be the harmonic function determined in the region T_0 with boundary C_0 by the boundary values $f_0(P)$, and $u_i(M)$ be the harmonic function determined in the region T_k with boundary $C_0 + C_1 + \cdots + C_k$, with boundary values $f_0(P), f_1(P), \cdots, f_k(P)$. Consider $\lim_{k=\infty} u_k(M)$. The functions u_k are bounded in their set.

If E is reducible, it must be reducible on each C_i. Hence it is possible to show (by taking each $\varrho_i^{(j)} < \delta_j/n_j$ where $\sum_1^\infty \delta_j$ is given with sum arbitrarily small, $< \delta$) that $|u_{k+p} - u_k|$ can be made uniformly small, n great enough, for all points on each given C_i or in T distant from E by $\geqq \delta'$, where δ' is fixed but arbitrary. In particular then, the convergence of u_k to some function u is uniform in the neighborhood of any interior point of T, and $u(M)$ is harmonic in T.‡ Also $u(M)$ takes on the boundary values $f_i(P)$ at almost all points of each C_i. By means of this reasoning we can state the theorem:

* O. D. Kellogg, Proceedings of the National Academy of Sciences, vol. 12 (1926), p. 400. The set E above is not a set B. Compare also the theorem of Art. 50, below.

† For such a discussion of necessary and sufficient conditions we refer again to Bouligand's Mémorial. The reader will find references to work of Zaremba, Lebesgue, Bouligand, Wiener and Kellogg.

‡ Osgood, loc. cit., p. 652.

With the given hypotheses on T, if E is reducible, a function $u(M)$ bounded and harmonic in T takes on boundary values in the wide sense almost everywhere on each C_i. If these values are assigned arbitrarily, bounded in their set and integrable in the Lebesgue sense, there is one and only one function $u(M)$, bounded and harmonic in T, which takes on these functions as boundary values almost everywhere on each C_i in the narrow (and therefore in the wide) sense.

48. Remarks on necessary and sufficient conditions. Our general point of view, especially in the first six chapters, has been the discussion of necessary and sufficient conditions. In other words we have dealt with the characterization of certain classes of harmonic functions. One feels somehow that it is desirable to have the characterization as simple as possible. But does it mean anything to say that a necessary and sufficient condition is simple? Let us dwell on this point merely long enough to answer the question in the negative—at least, if the reader will agree that it is not merely a question of simplicity in language, or choice of the smallest possible number of words.

We may suggest that simplicity means availability for application. On this hypothesis the term requires opposite qualities for necessary and sufficient conditions. For what makes a necessary condition simple is that it gives as much information as possible about a class of functions, and what makes a sufficient condition simple is that it requires as little information as possible. In other words, if we are dealing with a class of harmonic functions, a sufficient condition is simple if it states as little as possible in order to make the function belong to the class, beyond the fact that the function is harmonic; and thereby uses the harmonic property as much as possible in the demonstration of it; on the other hand, a necessary condition states as much as possible beyond the fact that the function is harmonic. The terms are necessarily vague, but perhaps an example will make clear the point of view.

Consider a function $u(r, \theta)$ harmonic within a unit circle.

The necessary and sufficient condition of Bray and the author (see Chap. III) that such a function be given by a Poisson integral in terms of its boundary values is that there should be a sequence of values r_i, $r_{i+1} > r_i$ with $\lim r_i = 1$, such that the absolute continuity of the integral

$$\int u(r_i, \theta)\, d\theta$$

be uniform for all i. Noaillon's condition (see also Chap. III) is that $u(r, \theta)$ should converge in the mean, of order 1, to a summable function $f(\theta)$. Consider the sufficiency condition.

It will be shown in Art. 49 that for any sequence of summable functions which converges in the mean, of order 1, the absolute continuity of the integral is uniform throughout the sequence. On the other hand, the sequence of functions $f_i(\theta) = \text{const.} = N_i$, where $\lim N_i = \infty$, is a sequence which satisfies the condition of Bray and the author without satisfying the condition of Noaillon; i. e., aside from the condition of being harmonic, less information is required for the former condition than for the latter. It is evident, for instance, that any bounded harmonic function satisfies the uniform absolute continuity condition, but not that it satisfies that of convergence in the mean. Similar remarks may be made for other special cases. Hence in spite of an apparent complexity of statement, it is clear that, as a sufficient condition, that of uniform absolute continuity is more useful. Perhaps there is a place for some other sufficient condition, still more simple than either; one which requires less of the general theory of integration of a sequence of functions.

On the other hand, as a necessary condition, Noaillon's condition states a simple and useful fact that is not a bare consequence of the other, but yields additional information which the other furnishes only through interrogating again the properties of harmonic functions. Moreover Noaillon's condition suggests an interesting classification, which we shall pursue further.

49. Convergence in the mean of positive order less than one.* With reference to an interval (a, b) we say that $f_m(x) \geqq 0$ converges in the mean, of order ν, $\nu > 0$, if $(f_m(x))^\nu$ is summable, and if

$$(9) \qquad \lim_{n=\infty} \int_a^b |f_m(x) - f_k(x)|^\nu \, dx = 0, \qquad m, k \geqq n.$$

In fact, since f_m and f_k are not negative we have

$$|f_m(x) - f_k(x)|^\nu \leqq \{f_m(x)\}^\nu + \{f_k(x)\}^\nu,$$

and therefore the left hand member of the inequality represents a summable function of x.

We say that $f_m(x) \geqq 0$ converges in the mean, of order ν, $\nu > 0$, to $f(x)$ if $\{f_m(x)\}^\nu$ and $\{f(x)\}^\nu$ are summable, and if

$$(10) \qquad \lim_{n=\infty} \int_a^b |f(x) - f_n(x)|^\nu \, dx = 0.$$

On the basis of these definitions a number of simple theorems may be established. For this purpose a well known fact for the case $\nu = 2$ will be assumed, viz., if $f_m(x)$ converges in the mean, of order 2, there will be a subsequence $f_{m_i}(x)$ which converges to a function $f(x)$ as a true limit, almost everywhere.

Lemma 1. *If $f_m(x) \geqq 0$ is convergent in the mean, of order ν, $\{f_m(x)\}^{1/2}$ is convergent in the mean, of order 2ν.*

In fact,

$$f_m{}^\nu = \{f_m{}^{1/2}\}^{2\nu}$$

and evidently

$$|f_m - f_k| \geqq (f_m{}^{1/2} - f_k{}^{1/2})^2,$$

$$|f_m - f_k|^\nu \geqq (f_m{}^{1/2} - f_k{}^{1/2})^{2\nu},$$

from which the proposition follows.

Lemma 2. *If $f_m(x) \geqq 0$ is convergent in the mean, of order ν, it is convergent in the mean of order $\varkappa$, where $\varkappa$ is arbitrary, less than ν.*

* The theorems given below apply also to $\nu > 1$. They are to some extent proved in Hobson, *The theory of functions of a real variable*, vol. II, Cambridge (1926), p. 239 ff., but for convenience in presentation are attached here to the classical case, $\nu = 2$.

Evidently $f_m^{\varkappa}$ is summable, if f_m^{ν} is summable. We are given further that

$$\int_a^b |f_m - f_k|^{\nu}\, dx < \varepsilon, \qquad m, k \geqq n.$$

Denote by $E_{m,k}$ the set

$$E_{m,k} = E(|f_m - f_k|^{\nu} \geqq \varepsilon^{1/2}).$$

Then meas. $E_{m,k} \leqq \varepsilon^{1/2}$, and in $E_{m,k}$ we have

$$|f_m - f_k|^{\varkappa} \leqq 1 \text{ or } < |f_m - f_k|^{\nu}.$$

Hence

$$\int_{E_{m,k}} |f_m - f_k|^{\varkappa}\, dx < \int_{E_{m,k}} \{1 + |f_m - f_k|^{\nu}\}\, dx$$
$$< \varepsilon^{1/2} + \int_a^b |f_m - f_k|^{\nu}\, dx < \varepsilon^{1/2} + \varepsilon.$$

But in $CE_{m,k}$, $|f_m - f_k|^{\varkappa} < \varepsilon^{\varkappa/2\nu}$, and

$$\int_{CE_{m,k}} |f_m - f_k|^{\varkappa} < \varepsilon^{\varkappa/2\nu}\,(b - a).$$

Hence finally,

$$\int_a^b |f_m - f_k|^{\varkappa}\, dx < \varepsilon^{\varkappa/2\nu}\,(b - a) + \varepsilon^{1/2} + \varepsilon, \qquad m, k \geqq n,$$

which has the limit 0 as n becomes infinite.

THEOREM 1. *If $f_m(x) \geqq 0$ is convergent in the mean, of order ν, there is a subsequence $f_{m_i}(x)$ and a function $f(x)$ such that*

(11) $$\lim_{i=\infty} f_{m_i}(x) = f(x), \textit{ almost everywhere.}$$

If $\nu \geqq 2$, it follows from Lemma 2 that $f_m(x)$ is convergent in the mean, of order 2; and the theorem reduces to the classical one.

If $0 < \nu < 2$, there will be, by Lemma 1, an integer p great enough so that

$$(f_m)^{1/2^p} = \varphi_m$$

will be convergent in the mean of order $2^p \nu \geqq 2$. Hence there will be a subsequence φ_{m_i} and a function φ such that

$$\lim_{i=\infty} \varphi_{m_i} = \varphi, \text{ almost everywhere.}$$

We write then

$$f(x) = \varphi^{2^p},$$

and the theorem is proved.

LEMMA 3. *If $f_m(x) \geqq 0$ is convergent in the mean, of order ν, the absolute continuity of $\int f_m{}^\nu \, dx$ is uniform for all m.*

Let us assume the contrary. If the absolute continuity is not uniform, there will be an $\eta > 0$ such that no matter what δ is given, arbitrarily small, and what p_0 arbitrarily large, there will be a set E, meas. $E < \delta$, and a $p, p > p_0$, for which

$$\int_E f_p{}^\nu \, dx > \eta.$$

Choose now p_0 great enough so that

$$\int_a^b |f_p(x) - f_{p_0}(x)|^\nu \, dx < \eta/2^{\nu+1}, \qquad p \geqq p_0,$$

and take δ small enough so that

$$\int f_{p_0}{}^\nu \, dx < \eta/2^{\nu+1} \text{ when meas. } e < \delta.$$

This yields a contradiction for some $p > p_0$. In fact,

$$\begin{aligned} \int_E f_p{}^\nu \, dx &= \int_E |f_{p_0} + f_p - f_{p_0}|^\nu \, dx \\ &\leqq 2^\nu \int_E f_p{}^\nu \, dx + 2^\nu \int_E |f_p - f_{p_0}|^\nu \, dx < \eta, \end{aligned}$$

which for some $p > p_0$ contradicts the first inequality.

An application of de la Vallée Poussin's theorem of Chap. I to Lemma 3 yields at once the fact that f^ν is summable, and that

$$\lim_{m=\infty} \int_{x_1}^{x_2} f_{m_i}{}^\nu \, dx = \int_{x_1}^{x_2} f^\nu \, dx.$$

Also

$$|f_m - f|^\nu = |f_m - f_{m_i} + f_{m_i} - f|^\nu$$
$$\leqq 2^\nu \{|f_m - f_{m_i}|^\nu + |f_{m_i} - f|^\nu\}$$

and

$$\int_a^b |f_m - f|^\nu \, dx \leqq 2^\nu \varepsilon + \int_a^b |f_{m_i} - f|^\nu \, dx.$$

But the absolute continuity of $\int |f_{m_i} - f|^\nu \, dx$ is uniform, since

$$|f_{m_i} - f|^\nu \leqq 2^\nu (f_{m_i}^\nu + f^\nu),$$

and therefore

$$\lim_{i=\infty} \int_a^b |f_{m_i} - f|^\nu \, dx = 0.$$

Hence

$$\lim_{m=\infty} \int_a^b |f_m - f|^\nu \, dx = 0,$$

and $f_m(x)$ converges in the mean, of order ν, to $f(x)$.

There cannot be any function $F(x)$ which differs from $f(x)$ on a set of positive measure, and possesses the property (11) or (10). For

$$|F - f|^\nu \leqq 2^\nu \{|F - f_m|^\nu + |f - f_m|^\nu\}.$$

Hence we have the following theorem:

THEOREM 2. *If $f_m(x) \geqq 0$ converges in the mean, of order ν, then*

$$\lim_{i=\infty} \int_{x_1}^{x_2} f_{m_i}^\nu \, dx = \int_{x_1}^{x_2} f^\nu \, dx,$$

for the m_i of Theorem 1, and $f_m(x)$ converges in the mean, of order ν, to $f(x)$. The function $f(x)$, except on an arbitrary set of zero measure, is the only such limit function.

We have also

$$|f_m - f_k|^\nu \leqq 2^\nu \{|f - f_m|^\nu + |f - f_k|^\nu\},$$

so that we can state immediately the converse of Theorem 2:

THEOREM 3. *If $f_m(x) \geqq 0$ converges in the mean, of order ν, to $f(x)$, it converges in the mean, of order ν.*

On the basis of these theorems we may prove a proposition which separates sequences which converge in the mean from those whose integrals become infinite.

THEOREM 4. *Let $f_m(x)$ be a sequence of not negative functions which has the limit* 0 *for almost all x. Then if $\nu > 0$ exists so that $f_m(x)$ fails to converge in the mean, of order ν, there will be a subsequence $f_{m_i}(x)$ such that for any $\varkappa$, $\varkappa > \nu$, we shall have*

$$\lim_{i=\infty} \int_a^b f_{m_i}{}^{\varkappa}\, dx = \infty .$$

Assume the theorem to be false. There will then be a subsequence $f_{m_k}(x)$ such that

$$\int_a^b f_{m_k}{}^{\nu}\, dx \geqq A > 0, \qquad k = 1, 2, \cdots .$$

Given a decreasing sequence of constants δ_i with $\lim \delta_i = 0$, there will be a corresponding sequence of sets E_i, with meas. $E_i < \delta_i$, and a subsequence of the f_{m_k}, namely f_{m_i}, such that

$$\begin{cases} \displaystyle\int_{E_i} f_{m_i}{}^{\nu}\, dx > \eta, & \eta \text{ some constant } > 0, \\ (f_{m_i}(x))^{\nu} > \delta_i{}^{-1\,2}, & x \text{ in } E_i . \end{cases}$$

In fact, we know from de la Vallée Poussin's theorem that there is an $\eta' > 0$ and a sequence of sets E_i' such that for a certain subsequence f_{m_i} we shall have

$$\int_{E_i'} f_{m_i}{}^{\nu}\, dx > \eta', \qquad \text{meas. } E_i' < \delta_i .$$

But if E_i'' is the portion of E_i' for which $(f_{m_i}(x))^{\nu} \leqq \delta_i{}^{-1/2}$, we shall have

$$\int_{E_i''} f_{m_i}{}^{\nu}\, dx \leqq \int_{E_i'} \delta_i{}^{-1/2}\, dx \leqq \delta_i^{1/2} .$$

If we write then $E_i = E_i' - E_i''$, take $\eta < \eta'$ and begin the sequence m_i late enough, all the conditions will be satisfied.

If now ϱ is any positive number,

$$\int_{E_i} f_{m_i}{}^{\nu+\varrho}\, dx > \delta_i^{-\varrho/2\nu} \int_{E_i} f_{m_i}{}^{\nu}\, dx > \eta\, \delta_i^{-\varrho/2\nu},$$

so that the limit of the left hand member becomes infinite with i. But all the more

$$\lim_{i=\infty} \int_a^b f_{m_i}{}^{\nu+\varrho}\, dx = \infty,$$

and this is what was to be proved.

We may apply this theorem directly to the Poisson-Stieltjes integral in terms of the following proposition.

THEOREM 5. *If $u(r, \theta)$ is given inside the unit circle by the Poisson-Stieltjes integral*

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r^2)\, d\, F(\varphi)}{1+r^2-2r\cos(\varphi-\theta)}$$

with $F(\varphi)$ of limited variation, and $F(\varphi+2\pi) = F(\varphi) + F(2\pi) - F(0)$, then, if we write $f(\theta) = F'(\theta)$ where that derivative exists, we have

$$\lim_{r=1} \int_0^{2\pi} |f(\theta) - u(r, \theta)|^{\nu}\, d\theta = 0,$$

for all $\nu < 1$.

For if this equation were not satisfied, there would be, for some $\nu < 1$, a sequence r_i of values r such that $|f(\theta) - u(r_i, \theta)| = f_i(\theta)$ did not converge in the mean, of order ν. But $\lim f_i(\theta) = 0$ almost everywhere; hence by Theorem 4 there would be a subsequence r_j, with $\lim r_j = 1$, so that

$$\lim_{j=\infty} \int_0^{2\pi} |f(\theta) - u(r_j, \theta)|\, d\theta = \infty$$

since $1 > \nu$. Hence

$$\lim_{j=\infty} \int_0^{2\pi} |u(r_j, \theta)|\, d\theta = \infty,$$

which is impossible, since $u(r, \theta)$ is of class (i).

In other words, if $u(r, \theta)$ is given by a Poisson-Stieltjes integral, but not by a Poisson integral, $u(r, \theta)$ fails to converge in the mean, of order 1, by Noaillon's theorem, but converges in the mean, of all orders < 1.

50. Integro-differential equations of Bôcher type. In a study in which are considered the behaviour of a function in general, its values almost everywhere, the values of its integral, and so on, one may well ask if the point of view should not be broader still, and if instead of Laplace's equation itself, some integral equation, or some equation in average values should not more properly have been considered. The answer to this demand is that under very general conditions, any equation which seeks to express in more general language the physical idea behind Laplace's equation will imply Laplace's equation itself.* One may use properties that depend on the mean value theorem, even to the extent of a definition by "médiation spatiale" after Zaremba and Lebesgue one may generalize the Laplacian operator by using a second order limit, or by regarding the operator as a functional derivative, or one may, with Bôcher, replace the equation of Laplace by suitable integro-differential equations. The latter seems closest to physical concepts of the operator.

The following theorem is a generalization of that of Bôcher.

Consider a class of simple rectifiable curves in T, each with only a finite number of vertices and such that the integral

$$\int_s \frac{\cos nr|}{r}\, ds$$

is bounded for each curve, where $r = PP'$, the distance between two points on the curve. Let $u(M)$ be summable on every curve of this class, and let the partial derivatives of u in the x and y directions be summable over the interior of such curves, and further, let the equations

$$(12)\qquad \begin{aligned} \int_s u\, dy &= \int_\sigma \frac{\partial u}{\partial x}\, d\sigma, \\ \int_s u\, dx &= -\int_\sigma \frac{\partial u}{\partial y}\, d\sigma \end{aligned}$$

be valid.

* See G. Bouligand, Mémorial, loc. cit., Arts. 3, 4, 5, 13.

If the equation

$$\int_s \frac{\partial u}{\partial x} dy - \frac{\partial u}{\partial y} dx = 0 \tag{13}$$

holds for all s in T of the given class, the function u has merely unneccssary discontinuities at a set of points of superficial measure 0, *and when these are removed the function* $u(M)$ *becomes harmonic in* T.*

It is sufficient to use instead of the derivatives $\partial u / \partial x$ and $\partial u / \partial y$ the derivatives in any two directions α, β and define the derivatives in other directions vectorially. One may, in fact, merely assume that the quantity

$$D_\alpha u = \lim_{\sigma=0} \int_s u \, d\alpha'$$

(α' the direction $\pi/2$ in advance of α, σ a "regular" family),

is defined almost everywhere in T for two given directions α, β, and satisfies the equations corresponding to (12). Then $D_s u$ is determined for an arbitrary direction s, and is a vector; and equation (13) takes the form

$$\int_s D_n u \, ds = 0. \tag{14}$$

These latter considerations become important for the consideration of the natural generalization of Poisson's equation

* G. C. Evans, Rice Institute Pamphlets, vol. 7 (1920), p. 286. For formula (23), p. 282 and that on the bottom of p. 279 there should be required an additional hypothesis, say that $\partial v / \partial x$ and $\partial w / \partial y$ remain bounded. This does not affect the validity of the theorem of Art. 5.33 (although it requires an obvious modification in the first part of the proof) nor of the theorem given above.

The theorem given above is true if the equation (13) is required merely for circles, as G. Bouligand remarks, on the basis of a general theorem of Zaremba. In fact, the demonstration cited above uses nothing but circles, and does not really require *all* circles in T. The equation (13) may also accordingly be required merely for rectangles. Bôcher's theorem is for circles, but assumes continuity. These considerations are not so important if one regards the subject from a physical point of view [see Evans, loc. cit., p. 261, also Cambridge Colloquium of the American Mathematical Society, Part I, New York (1918), p. 81 footnote].

$$\int_s D_n u \, ds = F(s), \tag{15}$$

where $F(s)$ is an additive function of curves of limited variation, corresponding therefore to an arbitrary completely additive function of point sets in the plane,—i. e., to an arbitrary distribution of mass in the plane.* The solution of Laplace's equation is therefore the key to the solution of (15) and to other partial differential equations of elliptic type, their corresponding integro-differential equations, and their discontinuous boundary value problems.

On a different side, the study of harmonic functions is of course interpreted by the theory of functions of a complex variable. But the problem becomes sharpened by the presence of two conjugate harmonic functions. From this point of view the discontinuous boundary value problem is allied to the study of the singularities of a Taylor's series on the circumference of the circle of convergence, and the determination of a Taylor's series in terms of its singularities in the plane or on its Riemann surface.†

* G. C. Evans, Rice Institute Pamphlets, loc. cit.

† See Hadamard and Mandelbrojt, *La série de Taylor et son prolongement analytique*, Scientia 41, 2nd edition, Paris (1926).

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